ALEXANDER-CONWAY INVARIANTS OF TANGLES

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ABSTRACT. We consider an algebra of (classical or virtual) tangles over an ordered circuit operad and introduce Conway-type invariants of tangles which respect this algebraic structure. The resulting invariants contain both the coefficients of the Conway polynomial and the Milnor's μ-invariants of string links as partial cases. The extension of the Conway polynomial to virtual tangles satisfies the usual Conway skein relation and its coefficients are GPV finite type invariants. As a by-product, we also obtain a simple representation of the braid group which gives the Conway polynomial as a certain twisted trace.

1. INTRODUCTION

The Alexander polynomial $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$ of a link L in $\mathbb{R}^3$ is one the most celebrated and well-studied link invariants. A number of different definitions and approaches to $\Delta_L(t)$ are known (see e.g. [21, 24]) and it is related to a variety of interesting objects and constructions. The Alexander polynomial has reappeared time and again in all major developments in knot theory of the last decades: quantum invariants, finite type invariants, and, lately, the theory of knot Floer homology (see e.g. [17]).

In its original form, $\Delta_L(t)$ is only defined up to multiplication by powers of t. Its close relative, the Conway polynomial $\nabla(L) = \sum_n c_n(L) z^n \in \mathbb{Z}[z]$, is free of this indeterminacy. The Conway polynomial may be obtained from $\Delta_L(t)$ by a substitution $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and is completely determined by its normalization $\nabla(O) = 1$ on the unknot O and the Conway skein relation

$$\nabla(L_+) - \nabla(L_-) = \nabla(L_0),$$

for any triple L_+, L_- and L_0 of link diagrams, which look as shown in Figure 1 in a certain disk B and coincide outside this disk.
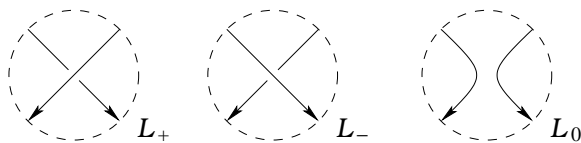

FIGURE 1. Conway skein triple

The importance of the Conway skein relation was realized after the appearance of the Jones polynomial, and led to the discovery of the HOMFLY polynomial [7].

The Alexander-Conway polynomial reappeared in the theory of quantum invariants (where it turned out to be related to the quantum supergroup $U_q(gl(1|1))$, and also to $U_q(sl(2))$ at roots of unity), and later in the theory of Vassiliev knot invariants, since all coefficients of $\nabla(L)$ are finite type invariants (see [2]), with $c_n(L)$ being an invariant of degree n.

Separate coefficients $c_n(L)$ of the Conway polynomial also attracted a lot of attention, with coefficients of low degrees been extensively studied. E.g., for knots, $c_2(L)$ is the Casson knot invariant. For 2-component links $c_1(L)$ is the linking number of two components, and $c_3(L)$ is the Sato-Levine invariant. For algebraically split 3-component links $c_2(L)$ is Milnor's triple linking number. More generally, the first non-vanishing coefficient of $\nabla(L)$ for n-component links is a certain combination of linking numbers [8]; for algebraically split links the first non-vanishing coefficient is a certain combination of Milnor triple linking numbers [13, 14, 23]. Coefficients $c_m(L)$ of a link L are related to those of a knot obtained by "banding together" components of L (see [13]).

After the development of the theory of virtual knots (see [11]), a number of attempts was made to extend the definition of the Alexander or the Conway polynomial to the virtual case using one of the original approaches for classical links. In particular, J. Sawollek [22] constructed a polynomial which, however, vanishes on classical links. One of the possible paths to pursue is that of the quantum invariants, or Fox differential calculus. This allows one to generalize the Alexander polynomial to (virtual or classical) tangles, but results in polynomials which are not invariant under the first Reidemeister move and do not satisfy the skein relation (e.g., a recent construction of Archibald and Bar-Natan [1]). We are unaware of any generalizations of the Alexander-Conway polynomial to virtual links or tangles, which would coincide with the original polynomial on classical links and satisfy the skein relation.

With this goal in mind, we follow the approach of Chmutov, Khouri and Rossi [3], who use a tautological state sum model of Jaeger [9] to deduce Gauss diagram formulas for the coefficients $c_n(L)$ of $\nabla(L)$. We extend and modify their construction to ordered tangles and provide a direct proof of the invariance under the Reidemeister moves. This enables us to extend the coefficients $c_n(L)$ to the virtual case as well. The resulting invariants are of finite type in the GPV sense [5] (and thus change under Kauffman's virtualization move [11]).

We consider ordered tangles as an algebra over an ordered circuit operad. Our Conway-type invariants of tangles respect this algebraic structure. This allows us to "break" any complicated tangle into elementary fragments, so all proofs and calculations may be done for elementary tangles. The resulting invariants contain the coefficients of the Conway polynomial of (long) links as a partial case. For string links we use a certain shifted ordering to obtain Milnor's triple and quadruple linking numbers. We conjecture that all Milnor's homotopy μ-invariants of string links may be obtained in this way. As a by-product, we also obtain a simple representation of the braid group, which gives the Conway polynomial as a certain twisted trace. Due to the Conway skein relation, this representation factors through the Hecke algebra.

The paper is organized in the following way. In Section 2 we review classical and virtual ordered tangles. Section 3 is dedicated to the ordered circuit operad and

its relation to ordered tangles. In Section 4 we introduce Conway-type invariants of tangles and then discuss their properties in Section 5.

2. Preliminaries

2.1. **Classical and virtual tangles.** Let B^3 be the unit 3-dimensional ball in $\mathbb{R}^3$. An *n-tangle* in B^3 is a collection S of n disjoint oriented intervals and some number of circles, properly embedded in B^3 in such a way, that the endpoints of each interval belong to the set $X = \{x_k\}_{k=1}^{2n}$, where x_k are some prescribed points on the boundary of B^3. For example, one may choose points x_k on the great circle $z = 0$, say, $x_k = (\exp(k\pi i/2n), 0)$ for odd k and $x_k = (\exp(-(k-1)\pi i/2n), 0)$ for even k.

Tangles are considered up to an oriented isotopy in B^3, fixed on the boundary. We will call embedded intervals and circles the *strings* and the *closed components* of a tangle, respectively. We will always assume that the only singularities of the projection of a tangle to the xy-plane are transversal double points. Such a projection, enhanced by an indication of an over/underpass in each double point, is called a *tangle diagram*. For technical reasons we will also often fix a base point (distinct from the endpoints of strings) on the boundary circle of the diagram. See Figure 2a.
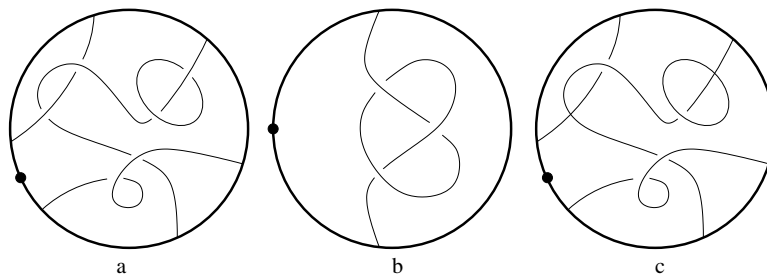

FIGURE 2. Classical and virtual tangle diagrams

Tangles generalize many objects, commonly considered in knot theory. In particular, tangles which have no strings are usual links in B. Tangles with one string and no closed components are *long knots*, see Figure 2b. Pure tangles without closed components are called *string links*. Here a tangle is *pure*, if the endpoints of k-th string, $k = 1, 2, \ldots, n$ are x_{2k-1} and x_{2k}. A particular example of a pure tangle is the unit tangle, with every pair x_{2k-1} and x_{2k} connected by an interval. Braids are tangles such that each tangle component intersects every plane $y = c$, $c \in [0, 1]$ in at most one point.

Virtual tangles present a useful generalization of tangles in the framework of the virtual knot theory [11]. In addition to usual crossings of a tangle diagram, one considers a new – virtual – type of crossings. We will draw virtual crossings as double points without any indication of the over- or underpass, see Figure 2c. Virtual tangle diagrams are considered up to the classical Reidemeister moves (see Figure 3) together with an additional set of virtual moves, shown in Figure 4.

Throughout the paper we will assume that all strings and closed components are oriented. Also, further we will consider only tangles with at least one string. To obtain invariants of closed classical links, we will pick one of the components, cut
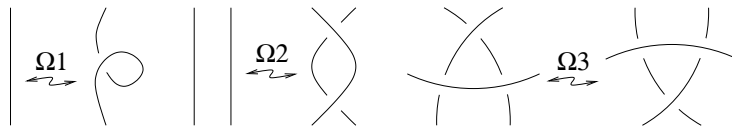
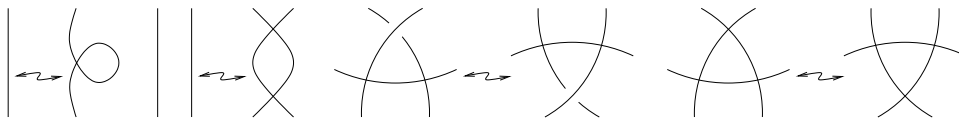FIGURE 3.   Classical Reidemeister moves

FIGURE 4.   Virtual Reidemeister moves

it open, and consider the resulting tangle with one string. We will then argue that the result does not depend on the choice. All our constructions will work both for classical and for virtual tangles.

2.2. **Ordered tangles.** Let T be a tangle with n strings. Since we assume that all strings are oriented, we can distinguish two endpoints of a string: its *input* (or source), and its *output* (or target). Denote by $\partial^- T$ and $\partial^+ T$ sets of inputs and outputs of all strings of T, respectively. A tangle T is *ordered*, if the set $X = \cup_{i=1}^{2n} x_i = \partial^- T \cup \partial^+ T$ is numbered by a collection $\{j_1 < j_2 < \cdots < j_{2n}\}$ of integers, so that each input is numbered by j_{2k-1}, and each output is numbered by j_{2k} for some $k = 1, 2 \ldots, n$. See Figure 5.
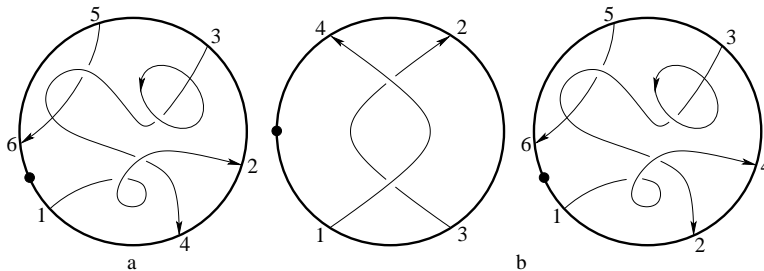

FIGURE 5.   Coherent and non-coherent tangle orderings

Orderings by different sets of integers which are related by a monotone map will be considered equivalent. Note that closed components do not appear in the definition of an ordering. Thus ordered closed links are usual links with no additional data, and 1-string tangles (in particular, long knots) have a unique ordering. We will call an ordering of a tangle T *coherent*, if for every $k = 1, 2, \ldots, n$ a string with the input numbered by j_{2k-1} has its output ordered by j_{2k}, see Figure 5a. For pure tangles (in particular string links) the standard ordering, such that x_k is labeled by k, is coherent.

3. CIRCUIT OPERAD AND TANGLES

3.1. **Circuit diagrams.** A circuit operad [1] is a modification of a planar tangles operad [10], adjusted for virtual tangles instead of classical tangles. It may be

useful to look at Figure 6 for some examples of circuit diagrams before reading the
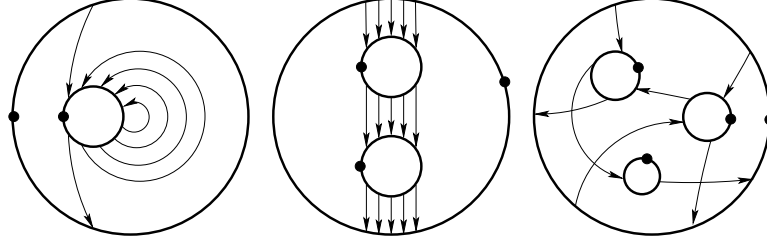formal definition below.

FIGURE 6. "Trace", "composition", and general circuit diagrams

An *n-circuit diagram* C, $n > 0$, is the unit disk D_0 in $\mathbb{R}^2$ with a (possibly empty)
collection of disjoint subdisks $D_1, D_2, \ldots, D_k$ in the interior of D_0. The boundary
∂D_0 of the disk D_0 is called the *output* of C, and the union of boundaries ∂D_i of D_i,
$i = 1, 2, \ldots, k$ is the *input* of C. Each disk D_i has an even number $2n_i$ of distinct
marked points on its boundary (with $n = n_0$). Each boundary circle ∂D_i, $i =
0, 1, \ldots, k$ is based, i.e., equipped with a base point distinct from the marked points;
we will denote it by $*_i$. The set of all marked points is equipped with a matching,
i.e., is split in $N = n_0 + n_1 + \cdots + n_k$ disjoint pairs. A pair is called *external*, if at least
one of its points lie on the output D_0, and is *internal* otherwise. It is convenient
to think about each pair as a simple path in the complement $D \smallsetminus \cup_{i=1}^{k} D_k$ of the
internal disks, connecting the corresponding marked points. These connections
represent only the matching of marked points, but not the actual path. Actual
paths, as well as their intersections, are irrelevant, so we will treat these intersections
as virtual crossings. Circuit diagrams are considered up to orientation-preserving
diffeomorphisms. If paths in a circuit diagram may be realized without intersections
in $D \smallsetminus \cup_{i=1}^{k} D_k$, we recover planar tangle diagrams of [10].

Two circuit diagrams C and C' with the appropriate number of boundary points
may be composed into a new circuit diagram $C \circ_i C'$ as follows. Isotope C' so that
the output $\partial D'_0$ of C', together with the set of marked points and the base point,
coincides with the input D_i of C. Glue C' into the internal disk D_i of C (smoothing
near the marked points so that the paths of C and C' meet smoothly) and remove
the common boundary. See Figure 7. This composition defines a structure of a
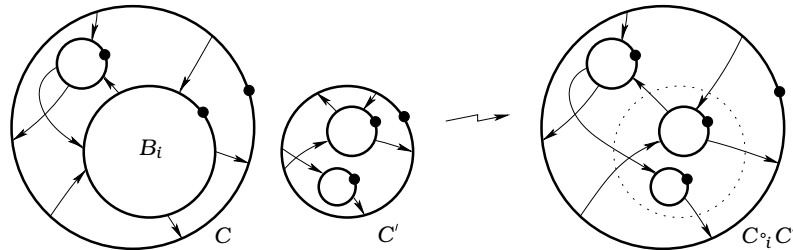colored operad on circuit diagrams.

FIGURE 7. Composing circuits

Further we will consider ordered oriented circuits. A circuit diagram is *oriented*, if all paths connecting pairs of marked points are oriented. In other words, each matched pair e of marked points is ordered: $(s(e), t(e))$ – the source and the target of e, respectively. The number of inputs and outputs on each circle ∂D_i are required to be equal (and thus equal to n_i). See Figures 6, 7. Compositions of oriented tangles should respect orientations of the paths.

3.2. **Ordered circuit diagrams.** An oriented circuit is *ordered*, if the set of paths is ordered, i.e., the set of all pairs of matched marked points is ordered: $(s_1, t_1), \ldots, (s_N, t_N)$. We assume that two following conditions hold. Firstly, points s_1 and t_N should be on the output ∂D_0 of C. Secondly, both points t_i, s_{i+1}, $i = 1, 2, \ldots, N - 1$ should be on the boundary of the same disk of C. See the rightmost picture of Figure 8.

An ordering of all external paths in C is called an *external ordering*, and an ordering of all internal paths in C is called an *internal ordering* of C. Given a (complete) ordering of C, its restriction to external and internal paths defines an external and an internal orderings of C, respectively. Vice versa, given both an internal and an external ordering of C, one may construct different complete orderings of C as shuffles of these two partial orderings, see Figure 8.

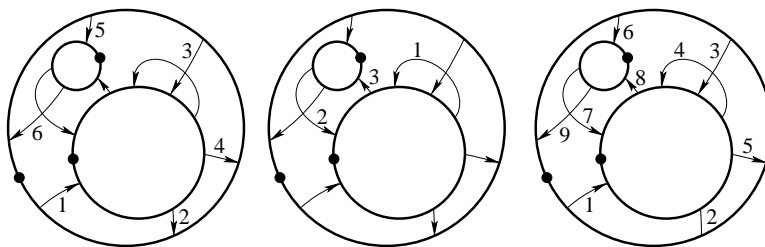

FIGURE 8.   External, internal, and complete orderings of a circuit

To define a composition $C \circ_i C'$ of ordered circuits C and C', we require the following compatibility of orderings. A *circuit* of length l in a circuit diagram C is a sequence $(s_j, t_j), \ldots, (s_{j+l}, t_{j+l})$ of paths, such that both endpoints s_j and t_{j+l} lie on the output ∂D_0, while the rest of the endpoints in this sequence lie on the inputs $\cup_{i=1}^{k} \partial D_i$. For example, an ordered circuit diagram in Figure 8 contains three circuits of lengths 2, 3, and 4, respectively. If a sequential pair t_i and s_{i+1} of marked points on ∂D_i are identified with a pair s'_j and t'_{j+l} of marked points on $\partial D'_0$, then we require that s'_j and t'_{j+l} are endpoints of a circuit in C'. The ordering on $C \circ C'$ is then induced from orderings on C and C'.

3.3. **Inserting tangles in circuit diagrams.** Given an n-circuit diagram C with k inputs and a k-tuple of tangles $T_1, \ldots, T_k$, we may create a new tangle $C(T_1, \ldots, T_k)$ if the data on the boundaries match. Isotope an n_i tangle T_i to such a position, that its boundary circle coincides with the input circle ∂D_i of C and the endpoints of strings in T_i coincide with the marked points on ∂D_i (if C and T_i are oriented, we also require a match of orientations). Glue T_i into the internal disk D_i of C along the boundary, removing common boundary circles and thinking about paths in the complement $D \setminus \cup_{i=1}^{k} D_k$ as a virtual tangle diagram. Doing this for all disks D_i, we obtain a new virtual n_0-tangle $C(T_1, \ldots, T_k)$ in the disk D_0. See Figure 9.
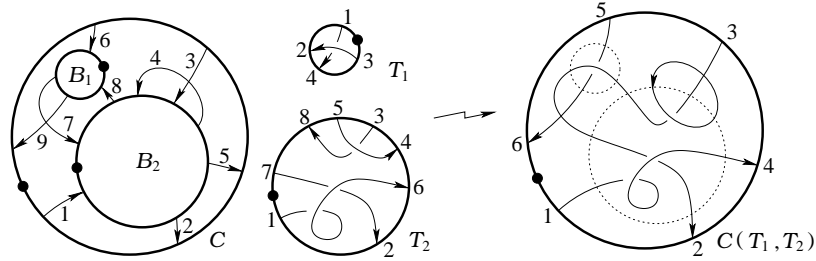
FIGURE 9. Inserting tangles in a circuit

Note that in order to define an ordering of the resulting tangle $C(T_1, \ldots, T_k)$, we need to fix only an external orientation on C; T_i's need not to be ordered. Moreover, if C has a (complete) ordering, it induces an ordering on each tangle T_i. Indeed, an ordering of C defines a numbering $i_1, o_1, \ldots, i_N, o_N$ of the set of all marked points, which induces an ordering of marked points on the boundary circle ∂D_i, and thus an ordering of T_i. If all T_i's and C are ordered, for the composition to be defined we require that this induced ordering should coincide with that of T_i. See Figure 9.

This operation gives to virtual tangles a structure of an algebra over the circuit operad, similar to the usual case of classical tangles as an algebra over the planar operad.

4. INVARIANTS OF ORDERED TANGLES

4.1. **States of tangle diagrams.** Let D be an ordered tangle diagram. An n-*state* of D is a collection of n crossings of D. A state S of D defines a new tangle diagram $D(S)$, obtained from D by smoothing all crossings of S respecting the orientation, see Figure 10a. The smoothed diagram inherits an ordering from D. We will say that the state S is *coherent*, if $D(S)$ contains no closed components and the ordering of $D(S)$ is coherent (see Section 2.2), i.e. if both ends of each string numbered by j_{2i-1} and j_{2i} for some i. Suppose that S is coherent. As we follow $D(S)$ along the first string of $D(S)$ (starting from its input and ending in its output), then continue to the second string of $D(S)$ in the same fashion, etc., we pass a neighborhood of each smoothed crossing $s \in S$ twice. A (coherent) state S is *descending*, if we enter this neighborhood first time on the (former) overpass of D, and the second – on the underpass. See Figure 10b. The sign $\text{sign}(S)$ of S is defined as the product of signs (local writhe numbers) of all crossings in S.

Remark 4.1. Property of coherency of a state depends on the ordering of a diagram D. Given a state S such that $D(S)$ contains no closed components, there are $n!$ orderings of D (which differ by reorderings of the n strings of $D(S)$) for which S is coherent. To estimate the number of orderings for which such a state S is descending, construct a graph with vertices corresponding to components (both open and closed) of D, with two vertices connected by an edge if S contains a crossing between two corresponding components of D. If the number of connected components of this graph is c, there are at most $c!$ orderings of D for which S may be descending. In particular, if the graph is connected, there is at most one ordering for which S may be descending.
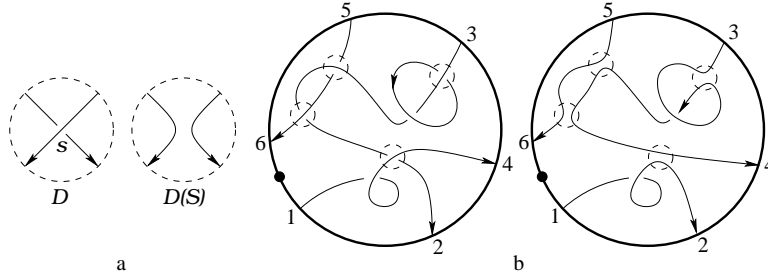
FIGURE 10.   Smoothing crossings which give a descending state

4.2. **Conway-type invariants of ordered tangles.** Denote by $\mathcal{S}_n(D)$ the set of all descending n-states of a diagram D. Define $c_n(D) \in \mathbb{Z}$ and $\nabla(D) \in \mathbb{Z}[z]$ by

$$c_n(D) = \sum_{S \in \mathcal{S}_n(D)} \text{sign}(S), \quad \nabla(D) = \sum_{n=0}^{\infty} c_n(D) z^n$$

In particular, $c_0(D) = 1$ if D has no closed components and the ordering of D is coherent (indeed, in this case the empty state $S = \emptyset$ is descending), and $c_0(D) = 0$ otherwise.

Example 4.2. For a trivial tangle diagram D_0 which consists of one straight string, the only descending state is trivial, thus $\nabla(D_0) = 1$. Also, an addition of a kink to a string of a tangle does not change the value of ∇ (since the new crossing cannot enter a coherent state). In particular, for a diagram D_0' obtained from D_0 by an addition of a small kink, we have $\nabla(D_0') = 1$. If a tangle diagram D_{split} is split – i.e., it can be subdivided into two non-empty non-intersecting parts – then there are no coherent (and thus no descending) states, thus $\nabla(D_{split}) = 0$.

Example 4.3. Let D_a, D_b, D_c and D_d be the diagrams shown in Figure 11. The only descending states for D_a and D_b are trivial, so $\nabla(D_a) = \nabla(D_b) = 1$. For D_c there are no descending states, so $\nabla(D_c) = 0$. For D_d there are two descending 1-states, but their signs are opposite, so $\nabla(D_d) = z - z = 0$.
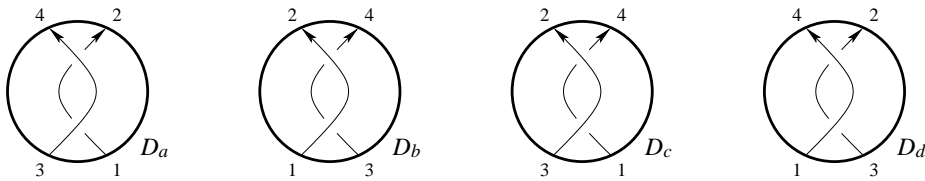

FIGURE 11.   Tangle diagrams with two strings

Example 4.4. For a diagram D_1 of a long Hopf link in Figure 12a there is only one descending state $\{2\} \in \mathcal{S}_1(D_1)$. Thus $\nabla(D_1) = \pm z$ (depending on the orientation of the closed component). For the diagram D_2 of a long trefoil in Figure 12b there is only one non-trivial descending state $\{2, 3\} \in \mathcal{S}_2(D_2)$, so $\nabla(D) = 1 + z^2$. For the diagram D_3 of a long iterated Hopf link in Figure 12c, there are three non-trivial descending states: $\{2\}, \{4\} \in \mathcal{S}_1(D)$, and $\{2, 3, 4\} \in \mathcal{S}_3(D)$. Thus $\nabla(D_3) = \pm(2z + z^3)$ (depending on the orientation of the closed component).
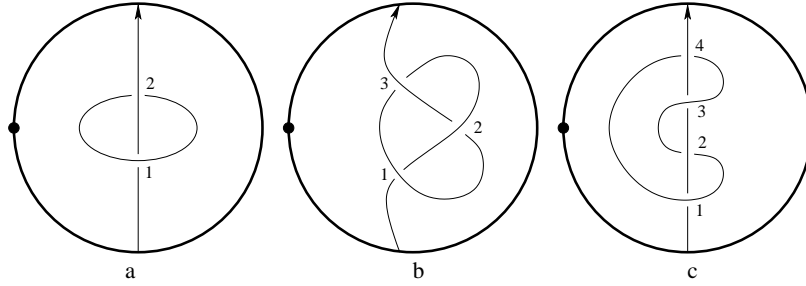
FIGURE 12. More tangle diagrams

Let C be on oriented circuit diagram with an external ordering. Let $T_1,\ldots,T_k$ be oriented tangles such that the composition $C(T_1,\ldots,T_k)$ is defined. Note that the external ordering of C makes $C(T_1,\ldots,T_k)$ into an ordered tangle. Now, suppose that a (complete) ordering or of C extends the given external ordering. As discussed in Section 3.3, or induces an ordering on each tangle T_i; denote the resulting ordered tangle by T_i^{or}. Directly from the definition of c_n we conclude that ∇ behaves multiplicatively under tangle compositions:

Lemma 4.5. *We have*

$$\nabla(C(T_1,\ldots,T_k)) = \sum_{or} \prod_{k=1}^{k} \nabla(T_i^{or}),$$

where the summation is over all orderings or of C, extending the initial external ordering.

Theorem 4.6. *Let D be a diagram of an ordered (classical or virtual) tangle T. Then $\nabla(T) = \nabla(D)$ defines an invariant of ordered tangles.*

Proof. Let us prove the claim by checking the invariance of $c_n(D)$ under the Reidemeister moves $\Omega 1 - \Omega 3$. The results of [19] imply that all oriented Reidemeister moves are generated by 4 moves shown in Figure 13 below.
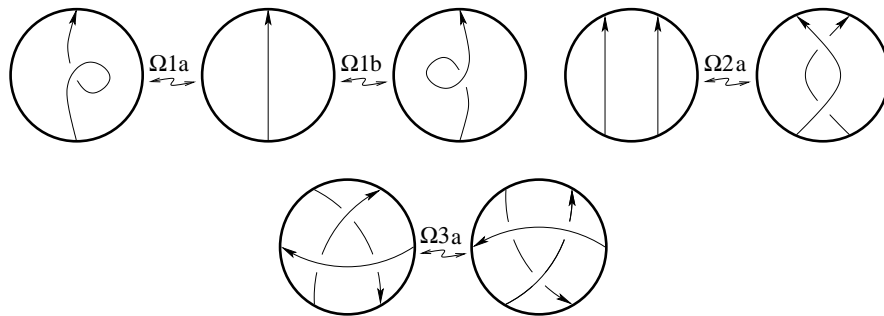

FIGURE 13. Elementary tangles related by Reidemeister moves

Due to Lemma 4.5, it suffices to verify the claim only on pairs of elementary tangle diagrams of Figure 13 for different orderings. Indeed, a pair of ordered tangles which are related by an oriented Reidemeister move in a certain disk B

can be presented as compositions $C(T_1, T_2)$ and $C(T_1', T_2)$ with T_1 and T_1' being elementary tangle diagrams inside the disk B, and T_2 being a tangle in a disk surrounding all other classical crossings, as illustrated in Figure 14.
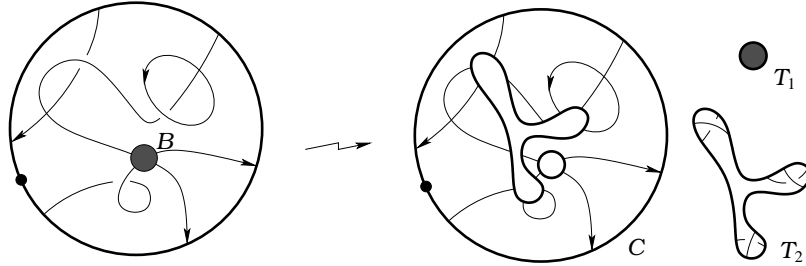

FIGURE 14.   Decomposing tangles into pieces

If a pair of ordered diagrams differ by $\Omega 1$, the equality $\nabla(T_1) = \nabla(T_1')$ follows from Example 4.2. A similar equality for $\Omega 2$ follows from Example 4.3.

Finally, if T and T' are two ordered tangle diagrams related by $\Omega 3$, there is an obvious bijective correspondence between all descending 0- and 1-states of T and T'; also, there are no descending 3-states. This implies the required equality for every ordering for which there are no descending 2-states of T and T'. But if for a certain ordering such states exist, they appear in pairs with opposite signs, and thus cancel out, see Figure 15.
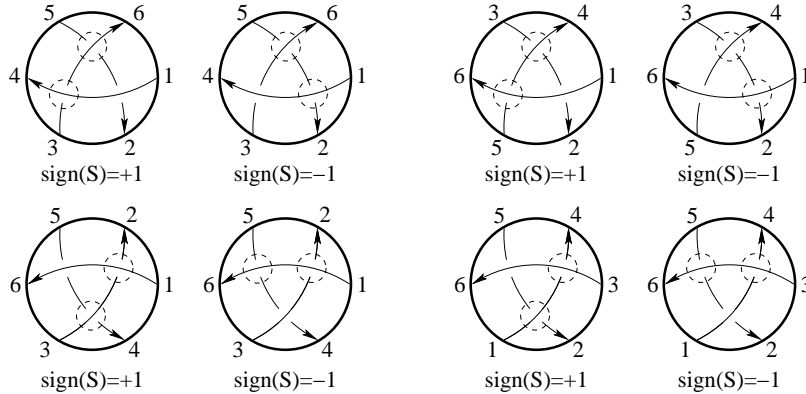
FIGURE 15.   Canceling pairs of descending states

□

5. Properties of the invariants

5.1. **Skein relation.** Our choice of notation ∇ for the invariant of Theorem 4.6 is explained in the following proposition:

Proposition 5.1. *Let T_+, T_- and T_0 be ordered tangles which look as shown in Figure 16a inside a disk and coincide (including orderings) outside this disk. Then the Conway skein relation holds:*

$$\nabla(T_+) - \nabla(T_-) = z\nabla(T_0)$$

FIGURE 16. Skein relation for tangles
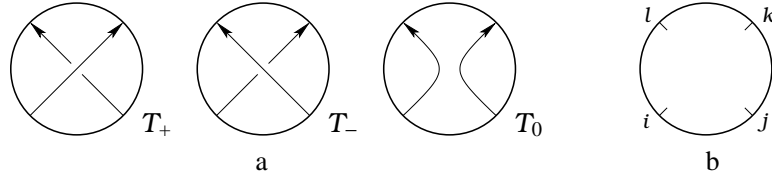
Proof. Due to the multiplicativity property of Lemma 4.5, it suffices to establish the equality for the three tangles of Figure 16a equipped with one of the four possible orderings shown in Figure 16b, with (i, j, k, l) being $(1, 3, 2, 4)$, $(1, 3, 4, 2)$, $(3, 1, 2, 4)$, or $(3, 1, 4, 2)$. The corresponding values of ∇ are shown in the table below.

$(ijkl)$	(1324)	(1342)	(3124)	(3142)
$\nabla(T_+)$	1	z	0	1
$\nabla(T_-)$	1	0	$-z$	1
$\nabla(T_0)$	0	1	1	0

$\square$

The above proposition, together with the normalization of Example 4.2 on split tangles and on the trivial 1-string tangle, imply the following corollary:

Corollary 5.2 (c.f. [3])**.** *Let T be a classical tangle with one string[1] and $m - 1$ closed components. Denote by $\bar{T}$ a closed m-component link, obtained from T by the braid-type closure. Then $\nabla(T)$ is the Conway polynomial of $\bar{T}$.*

5.2. **Relation to Milnor's μ-invariants.** For $n > 1$ invariants $c_n(T)$ include well-known Milnor's μ-invariants of link-homotopy [16] and their generalizations. In particular, let T be a tangle with n ordered strings and no closed components. Denote by T^σ the tangle T equipped with an ordering which is obtained from the standard one by a cyclic permutation $\sigma = (246 \ldots 2n)$, i.e. such that the source and the target of the k-th string are numbered by j_{2k-1} and j_{2k+2}, respectively, for $k = 1, \ldots, n - 1$, and the source and the target of the last string are numbered by j_{2n-1} and j_2. See Figure 17. Denote $\nu_{12\ldots n}(T) = c_{n-1}(T^\sigma)$.
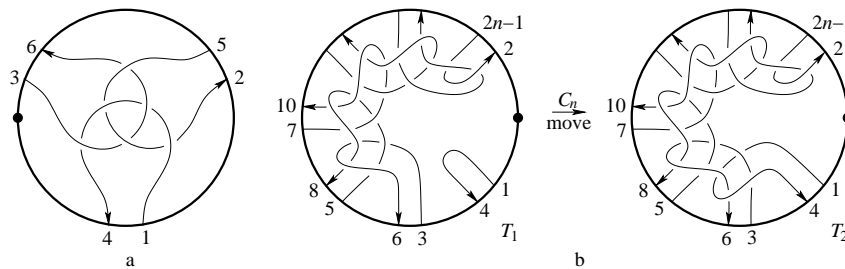

FIGURE 17. Tangles with a shifted ordering

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[1]Recall that for a tangle with one string there is only one possible ordering.

The invariant $\nu_{12\dots n}$ detects Goussarov-Habiro's C_n move [4, 6]: an easy calculation shows that for tangles T_1 and T_2 depicted in Figure 17b we have $\nu_{12\dots n}(T_1) = 0$, $\nu_{12\dots n}(T_2) = 1$.

Conjecture 5.3. *Modulo lower degree invariants, $\nu_{12\dots n}$ coincide with Milnor's link homotopy μ-invariant[2] $\mu_{2\dots n,1}$. In particular, if a closure $\hat{T}$ is a Brunnian link, so that n-th degree $\bar{\mu}$-invariants of $\hat{T}$ are well-defined, we have $\nu_{12\dots n}(T) = \bar{\mu}_{12\dots n}(\hat{T})$.*

Currently, the general structure of the lower degrees correction terms remains unclear. Here are some explicit formulas for low degree invariants, which directly follow from the results of [12, 18] (after a straightforward translation of formulas for $\nu_{123}(T)$ and $\nu_{1234}(T)$ to the language of Gauss diagrams):

Proposition 5.4. *Let T be a (possibly virtual) string link. Then*

$$\nu_{12}(T) = \mu_{2,1}(T) = \mathrm{lk}_{21}(T)\ , \quad \nu_{123}(T) = \mu_{23,1}(T) + \mathrm{lk}_{13}(T)\,\mathrm{lk}_{32}(T)$$

$$\nu_{1234}(T) = \mu_{234,1}(T) + \mu_{34,1}(T)\,\mathrm{lk}_{32}(T) + \mu_{24,1}(T)\,\mathrm{lk}_{43}(T) + \mu_{23,4}(T)\,\mathrm{lk}_{14}(T) +$$
$$\mathrm{lk}_{14}(T)\,\mathrm{lk}\,43(T)\,\mathrm{lk}_{32}(T)$$

Cases when $k \geq n$ are no less interesting. Here are some results for small values of n and k. Denote by $V_2(T)$ an invariant of 2-component string links introduced in [15], and let v_2 be the Casson knot invariant, i.e. the second coefficient of the Conway polynomial.

Proposition 5.5. *For a 2-string link T with a standard ordering we have $c_2(T) = V_2(T)$. This invariant is a splitting of the Casson knot invariant v_2, in the following sense. For a long knot K, denote by $K^2 = K \cup K'$ a 2-string link obtained from K by a "reverse doubling", i.e. by taking a pushed-off copy K' of K along a zero framing, and then reversing its orientation. Then $v_2(K) = -\frac{1}{2}V_2(K^2)$.*

Proof. The equality $c_2(T) = V_2(T)$ follows from equation 3.5 of [15]. Moreover, equation 2.2 of [15] implies that for any 2-string link T with strings T_1 and T_2 one has $V_2(T) = v_2(\bar{T}_{pl}) - v_2(T_1) - v_2(T_2)$, where $\bar{T}_{pl}$ is a plat closure of T. Apply this equality to $T = K^2$ and note that the plat closure $\bar{K^2}_{pl}$ of K^2 is the unknot, thus $V_2(K^2) = -v_2(K) - v_2(K')$. Also, v_2 is preserved under an orientation reversal, so $V_2(K^2) = -2v_2(K)$. $\square$

5.3. GPV finite type invariants. Finally, let us show that all invariants $c_n(T)$ are finite type invariants of virtual tangles in the sense of GPV [5]. Recall, that for virtual links there are two different theories of finite type invariants. Kauffman's theory of [11] is based on crossing changes similarly to the case of classical knots. GPV theory introduced in [5] is more restrictive in a sense that every GPV invariant is also a Kauffman invariant, but not vice versa. In GPV theory instead of crossing changes one uses a new operation, special for the virtual knot theory: crossing virtualization. Namely, given a real crossing in a diagram of a virtual link, we can convert it into a virtual crossing (resulting in a new virtual diagram).

Let D be a virtual tangle diagram with n marked and ordered real crossings d_i, $i = 1, \dots, n$. Given an n-tuple $I = \{i_1, \dots, i_n\} \in \{0,1\}^n$ of zeros and ones, denote by D_I a virtual tangle diagram obtained from D by virtualization of all crossings d_k with $i_k = 1$. Also, let $|I| = \sum_{k=1}^n i_k$. An invariant v of virtual tangles is called a

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[2]or rather, its generalization of [12] to (classical or virtual) tangles

GPV finite type invariant of degree less than n, if an alternating sum of its values on all diagrams D_I vanishes:

$$\sum_{I \in \{0,1\}^n} (-1)^{|I|} v(D_I) = 0$$

for all diagrams D and choices of crossings d_i. If v is of degree less than $n + 1$, but not less than n, we say that v is a GPV invariant of degree n. A restriction to classical tangles of a GPV invariant of degree n is a Vassiliev invariant of degree less than n.

Remark 5.6. An open (an highly non-trivial) conjecture is whether for long knots the opposite is true, i.e., whether any Vassiliev invariant of classical long knots can be extended to a GPV invariant. A positive solution would imply that Vassiliev invariants classify knots.

Proposition 5.7. *For any $n \in \mathbb{N}$, the invariant $c_n(T)$ is a GPV invariant of degree n.*

Proof. In [5] it is shown that any invariant given by an arrow formula, where all arrow diagrams contain at most n arrows, is a GPV invariant of degree at most n. A straightforward translation of this statement into the language of state sums of Sections 4.1-4.2 implies that any invariant defined by a state sum, where the states contain at most n crossings, is of degree at most n. $\square$

5.4. **Conway-type representation of a braid group.** If we restrict ourselves to braids instead of all n-tangles, we can apply results of the previous section to obtain a following representation of the braid group B_n on n strands.

Let $R_n = \mathbb{Z}S_n[z]$ be polynomials in one variable z with coefficients in the group ring $\mathbb{Z}S_n$ of the symmetric group S_n. Denote by σ_i and s_i, $i = 1, \ldots, n-1$ the standard generators of B_n and S_n respectively.

Proposition 5.8. *For $s \in S_n$, define*

$$\hat{\sigma}_i(s) = \begin{cases} s_i s & \text{if } s(i) < s(i+1) \\ s_i s + sz & \text{if } s(i) > s(i+1), \end{cases} \quad \hat{\sigma}_i^{-1}(s) = \begin{cases} s_i s & \text{if } s(i) > s(i+1) \\ s_i s - sz & \text{if } s(i) < s(i+1), \end{cases}$$

Set $\hat{\sigma}_i(sz^k) = \hat{\sigma}_i(s)z^k$ and extend it to R_n by linearity. This assignment defines an action of the braid group B_n on R_n. This representation satisfies $\hat{\sigma}_i(s) - \hat{\sigma}_i^{-1}(s) = zs$, thus factors through the Hecke algebra.

Proof. Use $s \in S_n$ to order inputs of all strings on the top of an elementary tangle σ_i by $2s_1 - 1, 2s_2 - 1, \ldots, 2s_n - 1$. The only two orderings of the outputs of σ_i for which there exist a descending state are $2s_1, \ldots, 2s_{i-1}, 2s_{i+1}, 2s_i, \ldots, 2s_n$ (for which the trivial state is descending) and $2s_1, \ldots, 2s_{i-1}, 2s_i, 2s_{i+1}, \ldots, 2s_n$ $s_i s$ (for which 1-state which consists of the only crossing of σ_i) is descending. See Figure 18. This immediately leads to the expression for $\hat{\sigma}_i(s)$. The expression for $\hat{\sigma}_i^{-1}(s)$ is obtained in the same way. $\square$

It would be interesting to identify this representation with (a quotient of) a known representation of the braid group.

The above representation may be used to recover the Conway polynomial as follows. For $s \in S_n$ and $\beta \in B_n$, denote by $\hat{\beta}[k](s)$ the coefficient of z^k in $\hat{\beta}(s)$;
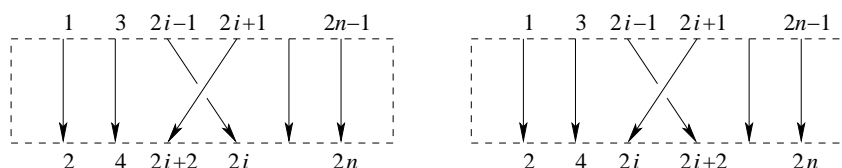
FIGURE 18.   Two possible orderings of an elementary tangle

this defines a linear operator on $\mathbb{Z}S_n$. Composing it with a shift $\tau(i) = i + 1$, $i = 1, \ldots, n - 1$, $\tau(n) = 1$ and taking the trace tr in $\mathbb{Z}S_n$, we define

$$c_k(\beta) = \mathrm{tr}(\tau \cdot \hat{\beta}[k])$$

Proposition 5.9. *Let L be a link, obtained as a closure of a braid β. Then $c_k(\beta)$ is the coefficient of z^k in the Conway-Alexander polynomial $\nabla(L)$ of L.*

Proof. Follows from Corollary 5.2.   □

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