COUNTING REAL CURVES WITH PASSAGE/TANGENCY CONDITIONS

SERGEI LANZAT AND MICHAEL POLYAK

Abstract. We study the following question: given a set $\mathcal{P}$ of $3d-2$ points and an immersed curve Γ in the real plane $\mathbb{R}^2$, all in general position, how many real rational plane curves of degree d pass through these points and are tangent to this curve. We count each such curve with a certain sign, and present an explicit formula for their algebraic number. This number is preserved under small regular homotopies of a pair $(\mathcal{P}, \Gamma)$ but jumps (in a well-controlled way) when in the process of homotopy we pass a certain singular discriminant. We discuss the relation of such enumerative problems with finite type invariants. Our approach is based on maps of configuration spaces and the intersection theory in the spirit of classical algebraic topology.

1. Introduction

1.1. **History.** A classical problem in enumerative geometry is the study of the number of certain algebraic curves of degree d passing through some number of points in the affine or projective plane. This question is not very interesting if we consider all curves of degree d, due to the fact that the set of such curves forms the projective space of dimension $\frac{1}{2}d(d+3)$, so the question of passing through points is simply a question of solving a system of linear equations. Thus, one usually asks this question about some families of algebraic curves of degree d, e.g., curves of a fixed genus g. In particular, there is an old question of determining the number N_d (resp. $N_d(\mathbb{R})$) of rational, i.e. genus $g=0$, curves of degree d passing through $3d-1$ points in general position in $\mathbb{CP}^2$ (resp. $\mathbb{RP}^2$). Here $3d-1$ is complex (resp. real) dimension of an algebraic variety of irreducible rational curves of degree d in $\mathbb{CP}^2$ (resp. $\mathbb{RP}^2$).

The numbers $N_1 = N_2 = N_1(\mathbb{R}) = N_2(\mathbb{R}) = 1$ go back to antiquity; $N_3 = 12$ was computed by J. Steiner in 1848. The late 19-th century was the golden era for enumerative geometry, and H.G. Zeuthen in 1873 could compute the number $N_4 = 620$. By then, the art of resolving enumerative problems had attained a very high degree of sophistication and, in fact, its

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foundations could no longer support it. Hilbert asked for rigorous foundation of enumerative geometry, including it as the 15-th problem in his list.

The 20-th century witnessed great advances in intersection theory. In the seventies and eighties, a lot of old enumerative problems were solved and many classical results were verified. However, the specific question of determining the numbers N_d turned out to be very difficult. In fact, in the eighties only one more of the numbers was unveiled: the number of quintics $N_5 = 87304$.

The revolution took place around 1994 when a connection between theoretical physics (string theory) and enumerative geometry was discovered. As a corollary, M. Kontsevich and Yu. Manin in [12] (see also [6]) gave a solution in terms of a recursive formula

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

But all these advances were done in the complex algebraic geometry. In the real case the situation is different. Until 2000 nothing was known about $N_d(\mathbb{R})$ for $d \geq 3$. In 2000 A. Degtyarev and V. Kharlamov [5] showed that $N_3(\mathbb{R})$ may be $8, 10$ or 12, depending on the configuration of $8 = 3 \cdot 3 - 1$ points in $\mathbb{RP}^2$. This result reflects a general problem of a real enumerative geometry: such numbers are usually not constant, but do depend on a configuration of geometrical objects. A natural way to overcome this difficulty is to try to assign some signs and multiplicities to objects in question so that the corresponding algebraic numbers remain constant. Already in the work of Degtyarev-Kharlamov one can see that one can assign certain multiplicities (signs) to real cubics passing through a given 8 points, so that the weighted sum of these cubics is independent on the configuration of points. In 2003 J.-Y. Welschinger [17] found a way to assign signs to real rational curves of any degree. Welschinger's sign w_C of a real rational curve C is defined as $w_C = (-1)^{m(C)}$, where $m(C)$ is the number of solitary points of C (called the mass of C). Welschinger's main theorem states that the corresponding weighted sum $W_d = \sum_C w_C$ of all curves passing through the given points is independent of the choice of points. The number W_d is called the Welschinger's invariant. In particular, $|W_d|$ gives a lower bound for the actual number $N_d(\mathbb{R})$ of real rational plane curves passing through any given set of $3d-1$ generic points. In the case of cubics ($d = 3$) from the Degtyarev-Kharlamov theorem one can see that $W_3 = 8$.

The question of passing through some number of points is the simplest one. The next step is to ask about the number of algebraic curves passing through some number of points and tangent to some given algebraic curves. In particular, in 1848, J. Steiner [16] asked how many conics are tangent to five given conics in $\mathbb{CP}^2$. The correct answer of 3264 was found by M. Chasles [4] in 1864. In 1984, W. Fulton asked how many of these conics can be real and in 1997, F. Ronga, A. Tognoli and Th. Vust [15] proved that all 3264 conics can be real. Another celebrated problem is due to Zeuthen. Given l lines and $k = d(d+3)/2 - 1$ points in $\mathbb{CP}^2$, the Zeuthen number $N_d(l)$ is the number of nonsingular complex algebraic curves of degree d passing through the k points and tangent to the l lines. It does not depend on the chosen generic configuration C of points and lines. If, however, points and lines are real, the corresponding number $N_d^{\mathbb{R}}(l, C)$ of real curves usually depends on the chosen configuration. For $l = 1$, it was shown by F. Ronga [14] that the real Zeuthen problem is maximal: there exists a configuration C such that $N_d^{\mathbb{R}}(1, C) = N_d(1)$. For $l = 2$ a similar maximality result was obtained by B. Bertrand [2] using Mikhalkin's tropical correspondence theorem. In the complex case in 1996 L. Caporaso and J. Harris found the recursive formulas in the spirit of M. Kontsevich for such tangency questions. In the real case that kind of questions of tangency is quite new and the serious development is just beginning. J.-Y. Welschinger in [18] considered curves in $\mathbb{RP}^2$ passing through a generic set of points and tangent to a non-oriented smooth simple zero-homologous curve. See Section 1.3 for a detailed discussion of Welschinger's results and a comparison with the present work.

We are interested to merge rigid algebro-geometric objects with flexible objects from smooth topology. We count algebraic curves in $\mathbb{R}^2$ that pass through a generic set of points and are tangent to an oriented immersed curve. In addition, we relate the dependence on a chosen configuration to the theory of finite type invariants.

1.2. **Motivation.** Let us start with a toy model: consider the case $d = 1$.

Let L be a set of lines in $\mathbb{R}^2$ passing through a fixed point p and tangent to a (generic) oriented immersed plane curve Γ. The problem is to introduce a sign ε_l for each such line $l \in L$ so, that the total algebraic number $N_1(p, \Gamma) = \sum_{l \in L} \varepsilon_l$ of lines does not change under homotopy of Γ in $\mathbb{R}^2 \smallsetminus p$. It is easy to guess such a sign rule. Indeed, under a deformation shown in Figure 1a, two new lines appear, so their contributions to $N_1(p, \Gamma)$ should cancel out. Thus, their signs should be opposite and we get the sign rule shown in Figure 1b. Note that only the orientation of Γ is used to define it; l is not oriented.

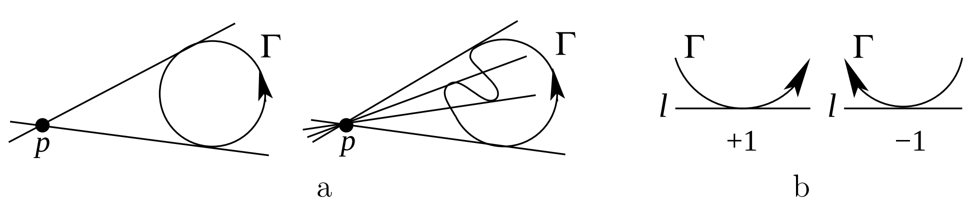
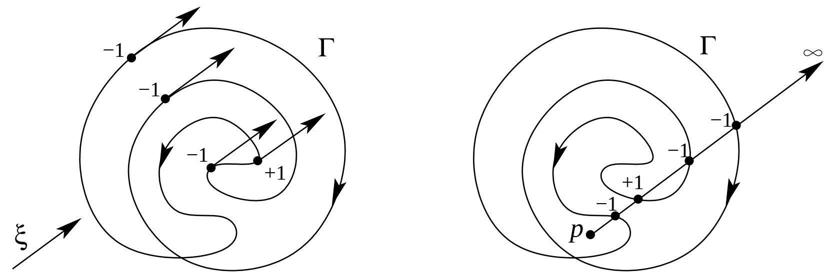


Figure 1. Counting lines with signs.

Suppose that p is close to infinity (i.e., lies in the unbounded region of $\mathbb{R}^2 \smallsetminus \Gamma$). In this case we get $N_1(p, \Gamma) = 2\,\mathrm{ind}(\Gamma)$, where $\mathrm{ind}(\Gamma)$ is the Whitney index (a.k.a. rotation number) of Γ, i.e. the number of turns made by the tangent vector as we pass once along Γ following the orientation. In other words, $\operatorname{ind}(\Gamma)$ equals to the degree of the Gauss map $G_\Gamma : \mathbb{S}^1 \to \mathbb{S}^1$ given by $G_\Gamma(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$, where $\gamma : \mathbb{S}^1 \looparrowright \mathbb{R}^2$ is a parametrization of Γ. Hence, the Whitney index can be calculated as an algebraic number of preimages of a regular value $\xi \in \mathbb{S}^1$ of the Gauss map G_Γ, see Figure 2a.



$\operatorname{ind}(\Gamma) = -1 - 1 - 1 + 1 = -2$ $\qquad$ $\operatorname{ind}_p(\Gamma) = -1 + 1 - 1 - 1 = -2$

a $\qquad$ b

FIGURE 2. Whitney index of Γ and an index of p w.r.t. to Γ.

While $N_1(p,\Gamma)$ is preserved under homotopies of Γ in $\mathbb{R}^2 \smallsetminus p$, it changes when Γ passes through p, see Figure 3.

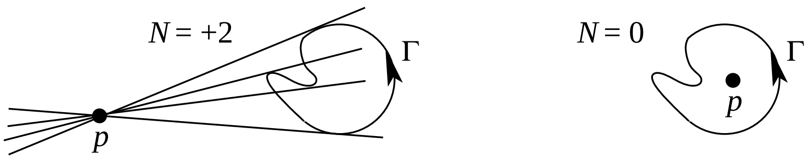


FIGURE 3. Counting lines with signs.

The compensating term is also easy to guess and we finally obtain

$$N_1(p,\Gamma) = 2\operatorname{ind}(\Gamma) - 2\operatorname{ind}_p(\Gamma)\ . \tag{1}$$

Here the index $\operatorname{ind}_p(\Gamma)$ of p w.r.t. Γ is the number of turns made by the vector connecting p to a point $x \in \Gamma$, as x passes once along Γ following the orientation. It may be computed as the intersection number $I([p,\infty],\Gamma;\mathbb{R}^2)$ of a 1-chain $[p,\infty]$ (i.e. an interval connecting p with a point near infinity of $\mathbb{R}^2$) with an oriented 1-cycle Γ in $\mathbb{R}^2$. See Figure 2b.

The appearance of $\operatorname{ind}(\Gamma)$ and $\operatorname{ind}_p(\Gamma)$ in the above formula comes as no surprise: in fact, these are the only invariants of the curve Γ under its homotopy in the class of immersions in $\mathbb{R} \smallsetminus p$. These are the simplest finite type invariants of plane curves, see [1].

In this simple example we see two main distinctive differences of real enumerative problems vs. complex problems of a similar passage/tangency type. Firstly, in the real case we are to count algebraic curves under the consideration with signs. Secondly, over $\mathbb{C}$ the answer is a number which does not depend on the relative position of the set of points and the curve Γ. Over $\mathbb{R}$, however, the answer depends on the configuration: it is preserved under small deformations of Γ and the set of points, but experiences certain (well-controlled) jumps when the configuration crosses certain singular discriminant in the process of homotopy. Thus, in the general case for similar enumerative problems we should not expect to get an answer as one number, but rather as a collection of numbers, depending on the relative configuration of points and the smooth curve.

Two main questions in this kind of problems are

1. How to find such sign rules, i.e. how to assign a certain sign to each algebraic curve under consideration, so that the total algebraic number is invariant under small deformations?
2. How does the singular discriminant looks like, and what is the explicit structure of the formula for the algebraic number of curves?

1.3. **Main results and the structure of the paper.** We count the algebraic number of real plane rational nodal curves of degree d passing through a given set of $3d-2$ generic points and tangent to a generic immersed curve in the plane $\mathbb{R}^2$. We get a number, which does not depend on a regular homotopy of the curve in a complement of a certain singular discriminant, see Subsection 2.1. As the curve passes through the discriminant, this number changes in a well-controlled way, so that it defines a finite type invariant of degree one, see Section 4. A mixture of rigid algebro-geometric objects with smooth topology gives to our problem a curious flavor, leading to a nice merging of features and techniques originating in both of these fields. In particular, this type of passage/tangency problems turns out to be intimately related to a theory of finite type invariants of plane curves, similarly to the toy case of $d = 1$ considered in Subsection 1.2 above.

For this we count rational nodal curves with signs and add certain correction terms, which come from degenerate cases of nodal, reducible and cuspidal curves. We use Welschinger's signs and show an easy way to produce new signs suitable for tangency questions. The technique of proofs uses the concept of configuration spaces and the intersection theory in the spirit of classical differential topology.

The question of passage/tangency conditions for real rational plane algebraic curves was considered earlier by J.-Y. Welschinger in [18]. He considered projective curves in $\mathbb{RP}^2$ passing through a generic set $\mathcal{P}$ and tangent to a non-oriented smooth simple zero-homologous curve Γ. In [18, Remark 4.3(3)] the author suggested the generalization to the case of a non-oriented smooth immersed curve Γ, which bounds an immersed disk; unfortunately, this formula does not extend to arbitrary immersed curves, e.g. to a figure-eight curve. There is a number of differences between [18] and the present work. Firstly, we consider oriented curves in $\mathbb{R}^2$ (thus adding orientations both to the curve and to the ambient manifold). Secondly, we consider immersed curves. Finally, in contrast with [18], where the author used 4-dimensional symplectic geometry and hard-core techniques from the theory of moduli spaces of pseudo-holomorphic curves, we use down to earth classical tools of differential topology. In this way we also get a clear geometric interpretation of Welschinger's number w_C as the orientation of a certain surface in $\mathbb{ST}^*\mathbb{R}^2$ (i.e. the manifold of oriented contact elements of the plane), which parameterizes real rational algebraic curves passing through $\mathcal{P}$.

The paper is organized in the following way. In Section 2 we introduce objects of our study, define signs of tangency, list the requirements of a general position, and formulate the main theorem. Section 3 is dedicated to the proofs. We interpret the desired number of curves as a certain intersection number; the main claim follows from different ways of its calculation. In Section 4 we discuss a relation of real enumerative geometry to finite type invariants.

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2. The Statement of the Main Result. Sketch of the Proof.

2.1. **Curves and points in general position.** Let $\mathcal{P} = \{p_1, \dots, p_{3d-2}\}$, $p_i \in \mathbb{R}^2$, $i = 1, \dots, 3d-2$ be a $(3d-2)$-tuple of (distinct) points in $\mathbb{R}^2$. Consider the following sets $\mathcal{C}_\mathcal{P}$, $\mathcal{T}_\mathcal{P}$, $\mathcal{R}_\mathcal{P}$ in $\mathbb{R}^2 \smallsetminus \mathcal{P}$ determined by $\mathcal{P}$:

- (i) We say that $p \in \mathcal{C}_\mathcal{P}$ (resp. $p \in \mathcal{T}_\mathcal{P}$), if there exists an irreducible rational curve of degree d, which passes through $\mathcal{P}$, has a cusp (resp. tacnode or a triple point) at p and whose remaining singularities are nodes in $\mathbb{R}^2 \smallsetminus \mathcal{P}$.
- (ii) We say that $p \in \mathcal{R}_\mathcal{P}$, if there exists a reducible curve $C_1 \cup C_2$ of degree d with $p \in C_1 \cap C_2$, which passes through $\mathcal{P}$ and such that C_1 and C_2 are irreducible rational nodal curves with nodes in $\mathbb{R}^2 \smallsetminus \mathcal{P}$, which intersect transversely at their non-singular points.

For $p \in \mathcal{P}$ denote by $\mathcal{D}(p)$ the set of all irreducible rational nodal curves of degree d, which pass through $\mathcal{P}$, have a crossing node at p and whose remaining nodes are in $\mathbb{R}^2 \smallsetminus \mathcal{P}$. Denote by $\mathcal{D}(\mathcal{P})$ the set $\mathcal{D}(p)$ together with curves listed in $(i) - (ii)$ above. Denote also $\mathfrak{S} := \mathcal{P} \cup \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$.

Suppose that the $(3d-2)$-tuple $\mathcal{P}$ is in general position. Explicitly, we will assume that the following conditions hold:

1. For any $k < d$, no $3k$ points from $\mathcal{P}$ lie on one rational curve of degree k.

2. The set $\mathfrak{S}$ is finite. Every point p in $\mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$ lies on exactly one curve from $\mathcal{D}(\mathcal{P})$. For every $p \in \mathcal{P}$ the set $\mathcal{D}(p)$ is finite.
3. All rational curves of degree d passing through $\mathcal{P}$ are either irreducible nodal, with nodes in $\mathbb{R}^2 \smallsetminus \mathfrak{S}$, or belong to $\mathcal{D}(\mathcal{P})$.

Now, let Γ be a generic immersed oriented curve in $\mathbb{R}^2$ in general position w.r.t. $\mathcal{P}$. Let us spell this requirement in more details. By a general position we mean that

4. The curve Γ is generically immersed, i.e., it is a smooth curve with a finite number of double points of transversal self-intersection as the only singularities.
5. The curve Γ intersects each of the curves from the set $\mathcal{D}(\mathcal{P})$ transversally, in points which do not belong to $\mathfrak{S}$.
6. Every irreducible rational nodal curve of degree d passing through $\mathcal{P}$ is tangent to Γ at most at one point, with the tangency of the first order. Every point of Γ is a point of tangency with at most one such a curve.

Define the singular discriminant Δ as the set of pairs $(\mathcal{P}, \Gamma)$ that violate the general position requirements listed above.

2.2. **Signs of points and curves.** Recall that the Welschinger's sign w_C of a rational curve C is defined as $w_C = (-1)^{m(C)}$, where $m(C)$ is the mass (i.e., the number of solitary points) of C. For each $p \in \mathfrak{S}$ we define ι_p by

$$\iota_p = \begin{cases} -W_d + 2 \cdot \sum_{C \in \mathcal{D}(p)} w_C & \text{if } p \in \mathcal{P}, \\ -w_C & \text{if } p \in \mathcal{C}_\mathcal{P}, \\ w_C & \text{if } p \in \mathcal{R}_\mathcal{P} \end{cases}$$

where C is (the unique) cuspidal or reducible curve of degree d passing through $\{p\} \cup \mathcal{P}$ for $p \in \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$.

Denote by $\mathcal{M}_d(\mathcal{P}, \Gamma)$ the set of real rational nodal curves passing through $\mathcal{P}$ and tangent to Γ. We fix the standard orientation $o_{\mathbb{R}^2}$ on the plane $\mathbb{R}^2$ once and for all. To each $C \in \mathcal{M}_d(\mathcal{P}, \Gamma)$ we assign a sign $\varepsilon_C = w_C \cdot \tau_C$, where τ_C is a sign of tangency of C with Γ, which is defined similarly to Subsection 1.2 as follows:

Let p be the point of tangency of Γ with C. For a sufficiently small radius r, C divides the disk $\mathbb{D}(p, r)$ centered at p into two parts. Since the tangency of Γ and C is of the first order, their quadratic approximations at this point p differ. Hence, $\Gamma \cap \mathbb{D}(p, r)$ belongs to the closure of one of the two parts of $\mathbb{D}(p, r) \smallsetminus C$. Let n be a normal vector to C at p, which looks into the closure of the part which contains Γ, and let t be the tangent vector t to Γ at p. Set $\tau_C = +1$ if the frame (t, n) defines the positive orientation $o_{\mathbb{R}^2}$ of $\mathbb{R}^2$, and $\tau_C = -1$ otherwise. See Figure 4 (compare also with Figure 1b).

Note that while the immersed curve Γ is oriented, the algebraic curve C is not, and we use just the orientation of Γ in order to define the sign τ_C.

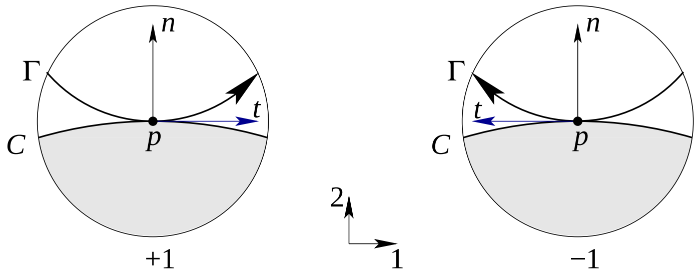

FIGURE 4. Signs of tangency τ_C.

2.3. **The statement of the main result.** Let $N_d(\mathcal{P},\Gamma)$ be the algebraic number

$$N_d(\mathcal{P},\Gamma) := \sum_{C\in\mathcal{M}_d(\mathcal{P},\Gamma)} \varepsilon_C$$

of real rational nodal curves passing through $\mathcal{P}$ and tangent to Γ. The main result of this work is the following

Theorem 2.1. *Let $\mathcal{P} = \{p_1,\ldots,p_{3d-2}\} \subseteq \mathbb{R}^2$ and Γ be an immersed oriented curve in $\mathbb{R}^2$, all in general position. Then*

$$N_d(\mathcal{P},\Gamma) = 2\left(W_d \cdot \operatorname{ind}(\Gamma) + \sum_{p\in\mathfrak{S}} \iota_p \cdot \operatorname{ind}_p(\Gamma)\right). \tag{2}$$

The number $N_d(\mathcal{P},\Gamma)$ is invariant under a regular homotopy of the pair $(\mathcal{P},\Gamma)$ in (each connected component of) the complement of the singular discriminant Δ.

2.4. **The case of cubics.** Degree $d = 3$ is the first case when all general difficulties appear. Namely, the number $N_3(\mathbb{R})$ is different from one and depends on a configuration of points, and curves may have cuspidal singularities. Although cubics have no tacnodes or triple points, these singularities do not contribute to (2), so are irrelevant for computation of $N_d(\mathcal{P},\Gamma)$.

There are three types of irreducible real rational cubics: cubics with one crossing node, cubics with one solitary node and cuspidal cubics with one cusp point. A reducible cubic is the union of a line and a conic. The number of curves in $\mathcal{D}(\mathcal{P})$ and points in $\mathcal{C}_\mathcal{P},\mathcal{R}_\mathcal{P}$ are bounded from above as follows. The number of points in $\mathcal{R}_\mathcal{P}$ is no more than $\binom{7}{2} = 21$. Due to [11], there are at most 24 cuspidal cubics passing through seven points in general position in $\mathbb{CP}^2$, hence $|\mathcal{C}_\mathcal{P}| \leq 24$. Also, there are no tacnodes or triple points, so $\mathcal{T}_\mathcal{P} = \varnothing$. From [17, Theorem 3.2] one can deduce that $|\mathcal{D}(p)| \in \{0,1\}$. Since $W_3 = 8$ and for a nodal cubic $m(C) = 0,1$ if C has a crossing or solitary node respectively, we get

$$\iota_p = \begin{cases} -8 + 2|\mathcal{D}(p)| & \text{if } p \in \mathcal{P}, \\ -1 & \text{if } p \in \mathcal{C}_{\mathcal{P}}, \\ +1 & \text{if } p \in \mathcal{R}_{\mathcal{P}}. \end{cases}$$

and Theorem 2.1 implies:

Corollary 2.2. *Let $\mathcal{P} = \{p_1, \dots p_7\} \subseteq \mathbb{R}^2$ and Γ be an immersed oriented curve in $\mathbb{R}^2$, all in general position. Then*

$$N_3(\mathcal{P}, \Gamma) = 2\left(8\,\mathrm{ind}(\Gamma) + \sum_{p\in\mathcal{P}} \iota_p \cdot \mathrm{ind}_p(\Gamma) - \sum_{p\in\mathcal{C}_{\mathcal{P}}} \mathrm{ind}_p(\Gamma) + \sum_{p\in\mathcal{R}_{\mathcal{P}}} \mathrm{ind}_p(\Gamma)\right)$$

2.5. **The main example.** Firstly, consider $\Gamma = T$, where $T = \partial\mathbb{D}(p, r)$ is a circle of infinitesimally small radius $0 < r << 1$ in $\mathbb{R}^2$, centered at p. Suppose that T is far from $\mathfrak{S}$, i.e, T is in the complement of some closed disk $\mathbb{D}^2$ which contains $\mathfrak{S}$. See Figure 5a.

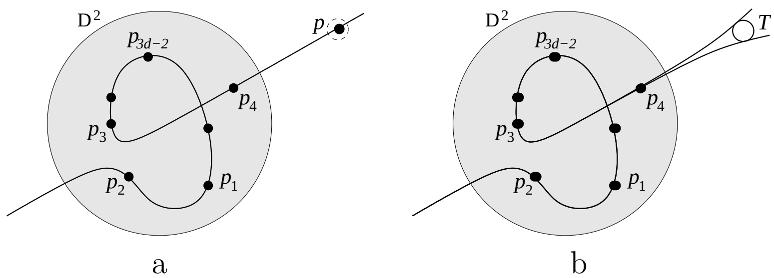

FIGURE 5. Changing a point into a circle.

Viewing T as a point p, i.e. taking the limit $r \to 0$, we get $3d - 1$ generic points $\{p\} \cup \mathcal{P}$ in the plane. We have W_d rational nodal curves of degree d passing through $\{p\} \cup \mathcal{P}$, counted with their Welschinger's signs. Thus the algebraic number $N_d(\mathcal{P}, \Gamma)$ of rational nodal curves passing through $\mathcal{P}$ and tangent to T counted with the sign ε_C is equal to $2W_d$. Indeed, each rational nodal curve passing through $\{p\} \cup \mathcal{P}$ gives 2 rational nodal curves passing through $\mathcal{P}$ and tangent to T, see Figure 5b. Moreover, from the definition of the sign τ_C we have that $\tau_C = +1$ for any $C \in \mathcal{M}_d(\mathcal{P}, \Gamma)$. Thus in this case

$$N_d(\mathcal{P}, \Gamma) = \sum_{C\in\mathcal{M}_d(\mathcal{P},\Gamma)} w_C \cdot \tau_C = 2\left(\sum_{C \text{ passes through } \{p\}\cup\mathcal{P}} w_C\right) = 2W_d\ .$$

Reparameterizing the circle T by $\mathbb{S}^1 \to \mathbb{S}^1,\ z \mapsto z^k,\ k \in \mathbb{Z}$ (and deforming it slightly into a general position) we get a curve denoted by $k \cdot T$ for which we have $\mathrm{ind}(k\cdot T)=k$ and

$$N_d(\mathcal{P},\Gamma)=2k\,W_d\ .$$

Since every immersed curve Γ is homotopic in the class of immersions in $\mathbb{R}^2$ to $k\cdot T$, where $k=\mathrm{ind}(\Gamma)$, we have that $N_d(\mathcal{P},\Gamma)=2W_d\cdot\mathrm{ind}(\Gamma)$ for a curve Γ lying in the complement of some closed disk of a sufficiently large radius, which contains $\mathfrak{S}$.

2.6. **The idea of the proof.** Consider a solid torus $M=\mathbb{D}^2\times\mathbb{S}^1$, where $\mathbb{D}^2$ is a sufficiently large closed disk containing $\mathfrak{S}$. We will show that the number $N_d(\mathcal{P},\Gamma)$ in Theorem 2.1 is the intersection number $I(L,\overline{\Sigma};M)$ of an oriented smooth curve L with a compactification $\overline{\Sigma}$ of an open two-dimensional surface Σ in M. The surface Σ is constructed as follows:
For each $p\in\mathbb{D}^2\smallsetminus\mathfrak{S}$, we use a contact element (line) of curves passing through $\{p\}\cup\mathcal{P}$ to get Σ as a lift of $\mathbb{D}^2\smallsetminus\mathfrak{S}$ into M. Lifting Γ into M in a similar way we get L. The Welschinger's sign w_C gives rise to the orientation on Σ and the orientation of Γ defines the orientation of L.
In order to define the intersection number, we compactify Σ to get a compact surface $\overline{\Sigma}$ with boundary. This is done by blowing up punctures $\mathfrak{S}$ on $\mathbb{D}^2$, i.e., we cut out a small open disk around each puncture and then we lift the remaining domain into M. Due to generality of a pair $(\mathcal{P},\Gamma)$, L transversally intersects $\overline{\Sigma}$ in a finite number of regular points of $\overline{\Sigma}$. Each point $(p,\xi)\in L\pitchfork\overline{\Sigma}$ corresponds to a curve passing through $\mathcal{P}$ and tangent to Γ. We prove that the local intersection number $I_{(p,\xi)}(L,\overline{\Sigma};M)$ equals to $\tau_C\cdot w_C$, and thus

$$N_d(\mathcal{P},\Gamma)=I(L,\overline{\Sigma};M).$$

Now to get the right hand side of the formula (2) we use the homological interpretation of the intersection number. We take $\Gamma':=\mathrm{ind}(\Gamma)\cdot T$ as in the main example, see Subsection 2.5, so Γ' is homotopic to Γ in the class of immersions. Hence $[\Gamma]-[\Gamma']=\partial K$ in $C_1(\mathbb{D}^2;\mathbb{Z})$ for some 2-chain $K\in C_2(\mathbb{D}^2;\mathbb{Z})$. Then for the lifts L' and $\mathcal{K}$ of Γ' and K, respectively, into M we have $[L]-[L']=\partial\mathcal{K}$ in $C_1(M;\mathbb{Z})$, and hence

$$I(L,\overline{\Sigma};M)=I(L',\overline{\Sigma};M)+I(\partial\mathcal{K},\overline{\Sigma};M).$$

From the main example we obtain $I(L',\overline{\Sigma};M)=2W_d\cdot\mathrm{ind}(\Gamma')=2W_d\cdot\mathrm{ind}(\Gamma)$. Finally, to complete the proof we show that

$$I(\partial\mathcal{K},\overline{\Sigma};M)=I(\mathcal{K},\partial\overline{\Sigma};M)=2\sum_{p\in\mathfrak{S}}\iota_p\cdot\mathrm{ind}_p(\Gamma).$$

Remark 2.3. *A simple way to visualize the surface Σ is to apply the above construction to the model example of Section 1.2. In this case $\mathfrak{S}$ consists of one point p and the contact element of any line passing through p is the line itself, so the surface is a helicoid, see Figure 9.*

3. The proof of the main result.

The manifold of oriented contact elements (directions) of the plane is $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$, the spherization of the cotangent bundle of the plane. We fix an orientation $o_{\mathbb{S}\mathrm{T}^*\mathbb{R}^2} = o_{\mathbb{R}^2} \times o_{\mathbb{S}^1}$ on $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$, where $o_{\mathbb{S}^1}$ is the standard counter-clockwise orientation on $\mathbb{S}^1$.

3.1. Construction of $M, \Sigma, \overline{\Sigma}, L$.

Construction of Σ. Consider a $(3d-2)$-tuple $\mathcal{P} = \{p_1, \dots, p_{3d-2}\}$ of points in $\mathbb{R}^2$ in general position. Recall that $\mathfrak{S} := \mathcal{P} \cup \mathcal{C}_{\mathcal{P}} \cup \mathcal{R}_{\mathcal{P}}$, see Subsection 2.1. Let $S := \mathbb{R}^2 \smallsetminus \mathfrak{S}$. Define

$$\Sigma := \left\{ (p,\xi) \in \mathbb{S}\mathrm{T}^*(\mathbb{R}^2 \smallsetminus \mathfrak{S}) \;\middle|\; \begin{array}{l} \text{there is a rational curve of degree } d \\ \text{passing through } \{p\} \cup \mathcal{P} \text{ and having } \xi \\ \text{as a tangent direction at a point } p \end{array} \right\}$$

Denote by $\pi : \Sigma \to S$ the natural projection $\pi((p,\xi)) = p$.

Proposition 3.1. *The set Σ is a (non-compact) immersed orientable two-dimensional surface in $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$.*

Proof. Consider an arbitrary $p \in S$ and choose a branch of a curve C_0 passing through p. Lifting the point p using the tangent direction ξ of this branch, we get a point in $\mathbb{P}\mathrm{T}^*\mathbb{R}^2$ which gives a pair $(p, \pm\xi)$ of points in the double covering $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$ of $\mathbb{P}\mathrm{T}^*\mathbb{R}^2$. Include the given curve C_0 into a smooth 1-parametric family C_t, $t \in (-\varepsilon, \varepsilon)$, $\varepsilon << 1$ of rational curves of degree d passing through $\mathcal{P}$. Since $p \notin \mathfrak{S}$, in a small neighborhood U of p the corresponding family of contact elements smoothly depends on the point of contact. The lift of this family of contact elements to Σ gives a smooth leaf $\Sigma(C_t)$ of Σ in a neighborhood of (p,ξ). The topological structure of this smooth leaf $\Sigma(C_t)$ depends on the type of the curve C_0.

If the initial curve C_0 is not cuspidal, the family C_t foliates U, so the projection π of $\Sigma(C_t)$ on U is a diffeomorphism. See Figure 6a. Note that the Welschinger's sign w of all curves C_t in the family is the same. This allows us to define a local orientation of $\Sigma(C_t)$ as follows. It suffices to define a continuous field ν normal to $\Sigma(C_t)$. Since $T_x\Sigma(C_t) \pitchfork T_x\mathbb{S}^1$ at each point $x \in \Sigma(C_t)$, such a normal vector ν_x is determined by its projection to $T_x\mathbb{S}^1$. Recall, that we have already fixed the orientation $o_{\mathbb{S}^1}$ on the fiber $\mathbb{S}^1$ of $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$. We set the direction of $T_x\mathbb{S}^1$-component of ν_x in the direction of $o_{\mathbb{S}^1}$ if $w = +1$, and opposite to this direction if $w = -1$.

If C_0 is cuspidal, both subfamilies $C^- = \{C_t | t \in (-\varepsilon, 0)\}$ and $C^+ = \{C_t | t \in (0, \varepsilon)\}$ foliate the same region of $U \smallsetminus C_0$, with Welschinger's sign $w^\pm$ of all curves in each subfamily $C^\pm$ being the same and $w^- = -w^+$, see [17, Proposition 2.16]. The corresponding leaf $\Sigma(C_t)$ of Σ has the structure of a fold, see Figure 6b. Two (open) sheets $\Sigma^\pm(C_t)$ of this fold correspond to the lift of contact elements of curves in the subfamilies $C^\pm$. Since curves in the

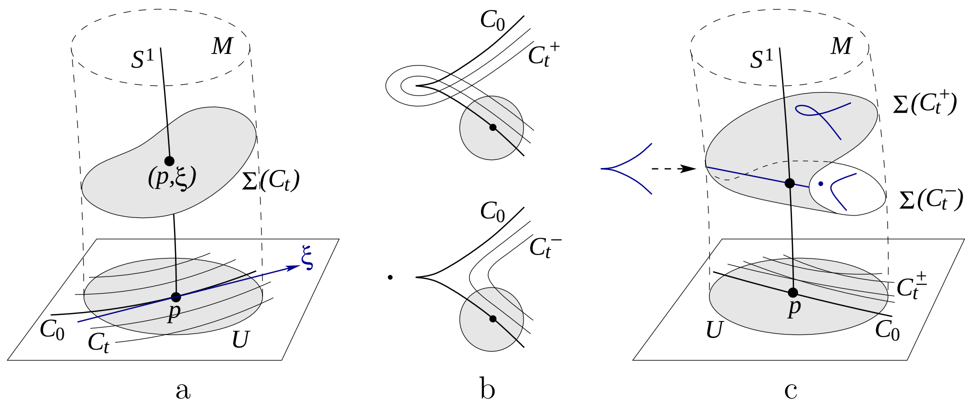

FIGURE 6. Smooth sheets and folds of Σ.

subfamilies $C^\pm$ have opposite Welschinger's signs $w^\pm$, we define the $T_x\mathbb{S}^1$-component of ν_x as above for $x \in \Sigma^\pm(C_t)$ and extend ν continuously (with $T_x\mathbb{S}^1$-component being zero) to the fold of $\Sigma(C_t)$ (i.e. the lift of C_0). □

The topological structure of Σ in a small neighborhood of (p, ξ) depends on the type of the point p. Namely, if p is generic then $3d-1$ points $\{p\} \cup \mathcal{P}$ are in general position and define a real general $\frac{(d-1)(d-2)}{2}$-dimensional space of curves of degree d that contains $N_d(\mathbb{R})$ irreducible rational nodal curves which intersect transversely in p. All of these curves pass through p with different tangent directions. Thus above a small neighborhood of a generic point p the surface Σ has a structure of a smooth $2N_d(\mathbb{R})$-covering. The same covering structure appears if p lies on a curve with a triple point or tacnode (but is different from a tacnode, $p \notin \mathcal{T}_\mathcal{P}$). Note that while the number $2N_d(\mathbb{R})$ of sheets over p depends on $\mathcal{P}$ and p, the number of sheets counted with their orientations (i.e., the local degree of $\pi : \Sigma \to S$ at a regular value p) does not depend on $\{p\} \cup \mathcal{P}$ and equals $2W_d$.

If some branches of curves which pass through $\{p\} \cup \mathcal{P}$ have the same tangent directions in p (in particular, if p is a tacnode), the corresponding lift has the structure of an open book, and sheets of the book come in pairs with each pair forming a smooth surface, see Figure 7a. The same open book structure appears if p lies on a reducible curve or a curve with a node at some $p_i \in \mathcal{P}$, see Figure 7b.

Remark 3.2. *We have to cut out points of $\mathfrak{S}$ from $\mathbb{R}^2$ in the construction of Σ above since Σ does not extend to an immersed surface over $\mathfrak{S}$. Indeed, if $p \in \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$, tangent directions to curves in 1-parametric families C_t used in the proof of Proposition 3.1 do not change smoothly in a neighborhood of p (see Figure 12b,c). If $p \in \mathcal{P}$, the obstacle is different: there are infinitely many tangent directions of curves passing through p. Note, that while points in $\mathcal{T}_\mathcal{P}$ are also singular points of curves in $\mathcal{D}(\mathcal{P})$, there is no need to cut them out from $\mathbb{R}^2$ since the corresponding tangent directions to curves in C_t*

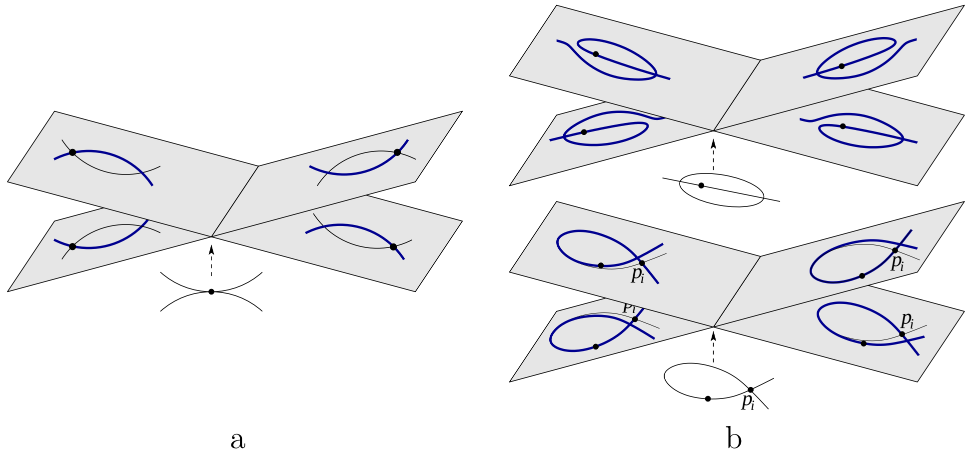

FIGURE 7. Branch points of Σ.

change smoothly in a neighborhood of a tacnode or a triple point, and the number of tangent directions in p is finite.

Compactification of $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$ and Σ. In order to use the intersection theory, we need to compactify both the open manifold $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$, and the non-compact surface Σ with punctures over $\mathfrak{S}$.

Let $\mathbb{D}^2 := \overline{\mathbb{D}(0,R)}, R >> 1$ be a closed disk in the plane $\mathbb{R}^2$, centered at the origin and of a sufficiently large radius, such that $\mathfrak{S} \subseteq \overline{\mathbb{D}(0,R/2)} \subseteq \mathbb{D}^2$. Define $M := \mathbb{S}\mathrm{T}^*\mathbb{D}^2 = \mathbb{D}^2 \times \mathbb{S}^1$.

Let us choose $0 < \delta << 1$ sufficiently small, such that

1. $\overline{\mathbb{D}(p,\delta)} \cap \overline{\mathbb{D}(q,\delta)} = \varnothing$ for all $p \neq q \in \mathfrak{S}$,
2. $\overline{\mathbb{D}(p,\delta)} \cap \Gamma = \overline{\mathbb{D}(p,\delta)} \cap \partial\mathbb{D}^2 = \varnothing$ for all $p \in \mathfrak{S}$,
3. $\overline{\mathbb{D}(p,\delta)}$ does not contain points but p of mutual intersections of all curves from $\mathcal{D}(\mathcal{P})$ for all $p \in \mathfrak{S}$,
4. $\partial\overline{\mathbb{D}(p,\delta)}$ intersects transversally every $C \in \mathcal{D}(\mathcal{P})$ for all $p \in \mathfrak{S}$. These intersections look as shown in Figure 8.

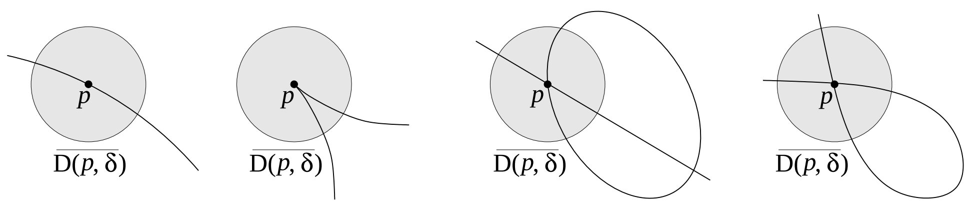

FIGURE 8. Intersections of blowup disks with cubics.

For each $p \in \mathfrak{S}$ we cut out the disk $\mathbb{D}(p,\delta)$ from $\mathbb{D}^2$ and define

$$\bar{S} := \mathbb{D}^2 \setminus \bigcup_{p\in\mathfrak{S}} \mathbb{D}(p,\delta), \qquad \overline{\Sigma} := \pi^{-1}(\bar{S}) = \Sigma \cap \left(\bar{S} \times \mathbb{S}^1\right).$$

For all $p \in \mathfrak{S}$ let $\sigma_p := \Sigma \cap (\partial\mathbb{D}(p,\delta) \times \mathbb{S}^1)$; it is the union of several smooth closed simple curves on Σ. We equip σ_p with the orientation induced from Σ. See Figure 9.

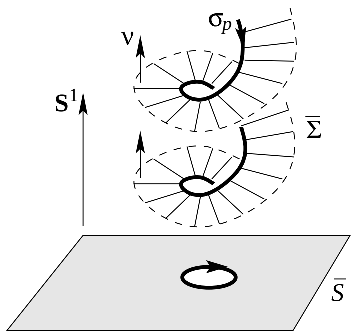

FIGURE 9. A compactification $\overline{\Sigma}$ of Σ.

Construction of L. Let $\Gamma = f(\mathbb{S}^1)$ be oriented immersed curve, where $f : \mathbb{S}^1 \looparrowright \mathbb{R}^2$ is an immersion. Choosing the unit tangent vector to $f(t)$ as the contact element, we get a lift L of Γ into $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$:

$$L := F(\mathbb{S}^1), \qquad F : \mathbb{S}^1 \hookrightarrow \mathbb{S}\mathrm{T}^*\mathbb{R}^2, \qquad t \mapsto \left(f(t), \frac{f'(t)}{\|f'(t)\|} \right).$$

It follows that L is an oriented closed one-dimensional submanifold of $\mathbb{S}\mathrm{T}^*\mathbb{R}^2$. If Γ is generic in the sense of Section 2.1, then L intersects Σ only in regular points (i.e., points such that $\Sigma \cap U$ is diffeomorphic to $\mathbb{R}^2$ for some neighborhood U in M).

3.2. **Two ways to calculate the intersection number** $I(L, \overline{\Sigma}; M)$**.** We will consider two different ways to calculate the intersection number $I(L, \overline{\Sigma}; M)$, which will correspond to the LHS and the RHS of equality (2).

The intersection number $I(L, \overline{\Sigma}; M)$ **via the algebraic number** N**.** Every point $(p,\xi) \in \overline{\Sigma} \cap L$ corresponds to an irreducible rational nodal curve $C(p,\xi)$, passing through $\mathcal{P}$ and tangent to Γ at the point p with the tangent direction ξ. Since $\mathcal{P}$ and Γ are in general position, we have that L and $\overline{\Sigma}$ intersect transversally, and since $\dim(L) = 1$, $\dim(\overline{\Sigma}) = 2$ and $\dim(M) = 3$, we have that $\dim(L \pitchfork \overline{\Sigma}) = 0$. So the number of points in $L \pitchfork \overline{\Sigma}$ is finite. Both L and $\overline{\Sigma}$ are oriented, as is M, hence the intersection number $I(L, \overline{\Sigma}; M)$ is well defined and we have

$$I(L, \overline{\Sigma}; M) = \sum_{(p,\xi)\in L\pitchfork\overline{\Sigma}} I_{(p,\xi)}(L, \overline{\Sigma}; M),$$

where $I_{(p,\xi)}(L, \overline{\Sigma}; M)$ is the local intersection number.

Proposition 3.3. *For every* $(p, \xi) \in L \pitchfork \overline{\Sigma}$ *we have*

$$I_{(p,\xi)}(L, \overline{\Sigma}; M) = \varepsilon_{C(p,\xi)},$$

where ε_C *is the sign of the curve* C*, see Subsection 2.2.*

Proof. The orientation of $T_{(p,\xi)}\overline{\Sigma}$ is defined by the Welschinger's sign $w_{C(p,\xi)}$. The curve L intersects $\overline{\Sigma}$ in the direction of the oriented fiber F iff $\tau_{C(p,\xi)} = +1$, see Figure 10. Hence the orientation of $T_{(p,\xi)}\overline{\Sigma} \oplus T_{(p,\xi)}L$ differs from the

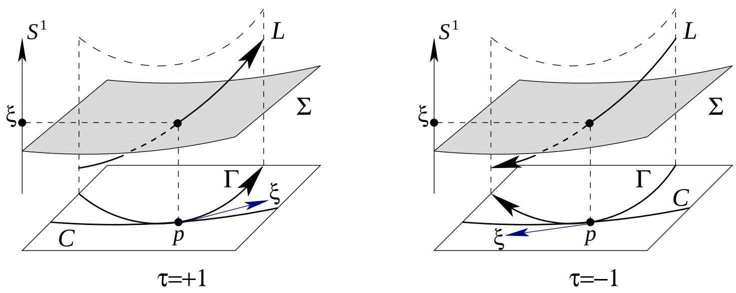

FIGURE 10. Intersection of L with Σ.

orientation $o_{\mathbb{ST}^*\mathbb{R}^2}\big|_M$ of M by the sign $\varepsilon_{C(p,\xi)} = w_{C(p,\xi)} \cdot \tau_{C(p,\xi)}$ and we get $I_{(p,\xi)}(L, \overline{\Sigma}; M) = \varepsilon_{C(p,\xi)}$. □

Corollary 3.4. *We have* $\quad N_d(\mathcal{P}, \Gamma) = \sum_{C \in \mathcal{M}_d(\mathcal{P},\Gamma)} \varepsilon_C = I(L, \overline{\Sigma}; M)$

The intersection number $I(L, \overline{\Sigma}; M)$ via a homological theory. Let us take $k \cdot T,\ k = \operatorname{ind}(\Gamma)$ as in Subsection 2.5 which is regularly homotopic to Γ in $\mathbb{D}^2$, and $h : \mathbb{S}^1 \times [0, 1] \to \mathbb{D}^2$ be a homotopy between $k \cdot T$ and Γ. Denote $\Gamma_t := h(\mathbb{S}^1 \times \{t\}),\ t \in [0, 1]$, so $\Gamma_0 = k \cdot T$ and $\Gamma_1 = \Gamma$. Denote by L_t a lift of Γ_t to M. Then $L' = L_0, L = L_1$ and a 2-chain $\mathcal{K} := \{L_t | t \in [0, 1]\} \in C_2(M; \mathbb{Z})$ realizes a homotopy between L' and L. We choose an orientation of $\mathcal{K}$ such that $\partial\mathcal{K} = [L] - [L']$. Because of the homotopy invariance of the intersection number, we may choose a special homotopy h as follows. For all $p \in \mathfrak{S}$ pick an open neighborhood U_p of $\overline{\mathbb{D}(p, \delta)}$ and a direction ξ_p transversal to tangent directions of all curves from $\mathcal{D}(\mathcal{P})$ at p. See Figure 11a. Now, choose the homotopy h so that for all $t \in [0, 1]$ with $\Gamma_t \cap U_p \neq \varnothing$, the fragment $\Gamma_t \cap U_p$ is close to a straight interval in the direction ξ_p. For such a homotopy the part $\mathcal{K} \cap (U_p \times \mathbb{S}^1)$ of $\mathcal{K}$ over U_p is almost flat, i.e., lies in a thin cylinder

$$\mathcal{K} \cap (U_p \times \mathbb{S}^1) \subseteq U_p \times (\xi_p - \varepsilon, \xi_p + \varepsilon),$$

for some small $0 < \varepsilon << 1$. See Figure 11b.

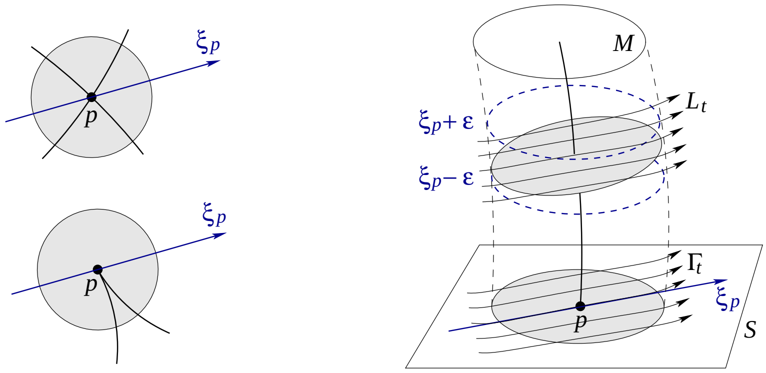

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FIGURE 11. A flat homotopy of Γ.

By the additivity of the intersection number and according to the calculations in the Subsection 2.5 we have

$$I(L,\overline{\Sigma};M) = I(L',\overline{\Sigma};M) + I(\partial\mathcal{K},\overline{\Sigma};M) = \\ = 2W_d\cdot\operatorname{ind}(k\cdot T) + I(\partial\mathcal{K},\overline{\Sigma};M) = 2W_d\cdot\operatorname{ind}(\Gamma) + I(\partial\mathcal{K},\overline{\Sigma};M)$$

It remains to compute $I(\partial\mathcal{K},\overline{\Sigma};M)$.

Lemma 3.5. *For $\overline{\Sigma},\mathcal{K},\sigma_p$ and L as before we have*

$$I(\partial\mathcal{K},\overline{\Sigma};M) = \sum_{p\in\mathfrak{S}} I(\mathcal{K},\sigma_p;M).$$

Proof. Recall that $I(\partial\mathcal{K},\overline{\Sigma};M) = I(\mathcal{K},\partial\overline{\Sigma};M)$. Now, as a 1-chain in M,

$$\partial\overline{\Sigma} = \partial\overline{\Sigma}\cap(\partial\mathbb{D}^2\times\mathbb{S}^1)) + \sum_{p\in\mathfrak{S}}\sigma_p.$$

Since $\mathcal{K}\cap(\partial\mathbb{D}^2\times\mathbb{S}^1) = \varnothing$, we get $I(\mathcal{K},\partial\overline{\Sigma};M) = \sum\limits_{p\in\mathfrak{S}} I(\mathcal{K},\sigma_p;M)$. □

The following proposition completes the proof of the main theorem.

Proposition 3.6. *For every $p\in\mathfrak{S}$ we have*

$$I(\mathcal{K},\sigma_p;M) = 2\iota_p\cdot\operatorname{ind}_p(\Gamma).$$

Proof. Firstly, recall that $\mathbb{S}\mathrm{T}^*\mathbb{D}^2\to\mathbb{P}\mathrm{T}^*\mathbb{D}^2$ is a 2-fold covering, so for every component of the lift of $\partial\mathbb{D}(p,\delta)$ to $\mathbb{P}\mathrm{T}^*\mathbb{D}^2$ there are two components in $\mathbb{S}\mathrm{T}^*\mathbb{D}^2$, which explains the coefficient 2 in the RHS. Secondly, note that $I(\mathcal{K},\sigma_p;M) = I(\mathcal{K}_p,\sigma_p;M)$ for every $p\in\mathfrak{S}$, where $\mathcal{K}_p := \mathcal{K}\cap(U_p\times\mathbb{S}^1)$.
In order to compute $I(\mathcal{K}_p,\sigma_p;M)$ we study the homology class $[\sigma_p]\in\mathrm{H}_1(M;\mathbb{Z})$ of σ_p. Since $\mathrm{H}_1(M;\mathbb{Z}) = \mathbb{Z}\langle[F]\rangle$, where $[F]$ is the class of the fiber, we conclude that $[\sigma_p] = k_p\cdot[F]$ for some $k_p\in\mathbb{Z}$. The number k_p is the degree $\deg G_p$ of the corresponding projection map $G_p:\sigma_p\to F$ to the fiber F of

M. It can be computed as the algebraic number of preimages $(G_p)^{-1}(\xi)$ of a regular value ξ. Each preimage $(q, \xi) \in \sigma_p$ is counted with its sign – the local degree $\deg_{(q,\xi)} G_p$ of G_p at (q, ξ).

Both preimages and their signs can be recovered from the projection $\pi : \sigma_p \to \partial\mathbb{D}(p, \delta)$. Indeed, a preimage $(q, \xi) \in \sigma_p$ corresponds to the point $q \in \partial\mathbb{D}(p, \delta)$ and the curve $C(q, \xi)$ passing through $q \cup \mathcal{P}$ and having a tangent direction ξ at q. Moreover, the orientation of $T_{(q,\xi)}\sigma_p$ is induced from that of Σ which, in turn, is defined by the Welschinger's sign $w_{C(q,\xi)}$. Thus the orientation of the projection $\pi : \sigma_p \to \partial\mathbb{D}(p, \delta)$ at q differs from the clockwise orientation on $\partial\mathbb{D}(p, \delta)$ by $w_{C(q,\xi)}$ (see Figure 9) . Therefore, the local degree $\deg_{(q,\xi)} G_p$ equals to $w_{C(q,\xi)} \cdot \rho_q$, where $\rho_q = 1$ (resp. $\rho_q = -1$) if the field of tangent directions of curves corresponding to σ_p rotates counterclockwise (resp. clockwise) w.r.t. ξ as we move clockwise along $\partial\mathbb{D}(p, \delta)$ in a neighborhood of q.

To find the corresponding curves $C(q, \xi)$, note that any curve passing through $q \cup \mathcal{P}$ for $q \in \partial\mathbb{D}(p, \delta)$ is obtained by a small deformation of some rational curve C_p of degree d in the following finite set:

(i) If $p \notin \mathcal{P}$, then C_p passes through $3d - 1$ points $p \cup \mathcal{P}$.
(ii) If $p \in \mathcal{P}$, then C_p passes through $3d - 2$ points $\mathcal{P}$ and either has a node at p, or has a tangent direction ξ at p.

Consider these cases separately using the standard methods of singularity theory.

Case 1: $p \notin \mathcal{P}$**.** If C_p is nodal, its small deformation is shown in Figure 12a. For sufficiently small δ, the corresponding tangent field is almost constant and for a generic choice of ξ there are no preimages.

If $C_p \in \mathcal{D}(\mathcal{P})$, its small deformations for $p \in \mathcal{C}_{\mathcal{P}}$, and $p \in \mathcal{R}_{\mathcal{P}}$ are shown in Figures 12b and 12c respectively.

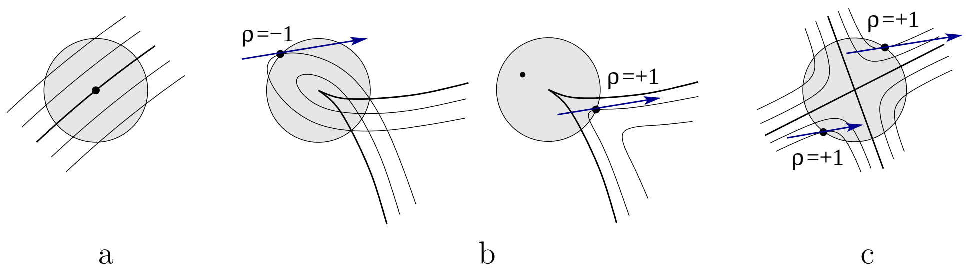

FIGURE 12. Counting preimages for $p \notin \mathcal{P}$.

For $p \in \mathcal{R}_{\mathcal{P}}$, there are two preimages, both with $\rho_q = 1$ and $w_{C(q,\xi)} = w_{C_p}$, see Figure 12c. Thus the local degree of each of these two preimages equals $w_{C_p} = \iota_p$, so $\deg G_p = 2\iota_p$.

For $p \in \mathcal{C}_{\mathcal{P}}$ there are also two preimages: one with $\rho_q = -1$ and $w_{C(q,\xi)} = w_{C_p}$, and the other with $\rho_q = 1$ and $w_{C(q,\xi)} = -w_{C_p}$, see Figure 12b. Thus the local degree of each of these two preimages equals $-w_{C_p} = \iota_p$ and $\deg G_p = 2\iota_p$.

Case 2: $p \in \mathcal{P}$**.** If C_p is nodal with node at p (i.e., $C_p \in \mathcal{D}(p)$), its small deformations are shown in Figure 13a. Again, for sufficiently small δ, the corresponding tangent fields are almost constant and for a generic choice of ξ there are no preimages.

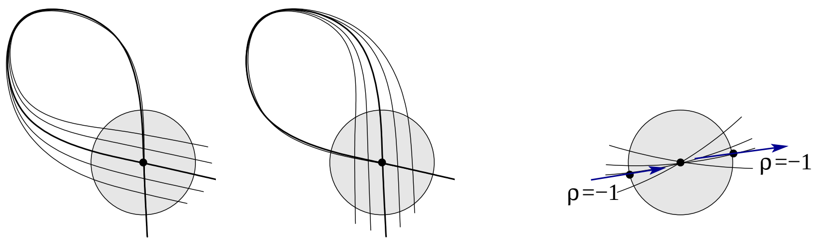

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FIGURE 13. Counting preimages for $p \in \mathcal{P}$.

If $C_p = C(p,\xi)$ is nodal with a tangent direction ξ at p, its small deformations are shown in Figure 13b. There are two preimages, both with $\rho_q = -1$ and $w_{C(q,\xi)} = w_{C(p,\xi)}$, see Figure 13b. Thus the local degree of each of these two preimages equals $-w_{C(p,\xi)}$ and each such curve $C(q,\xi)$ contributes $-2w_{C(p,\xi)}$ to $\deg G_p$. By [17, Proposition 3.4]

$$\sum_{C(p,\xi)} w_{C(p,\xi)} + 2 \sum_{C \in \mathcal{D}(p)} w_C = W_d \,,$$

where the first sum is over all nodal curves with a tangent direction ξ at p. Therefore, in this case we also get

$$\deg G_p = -2 \sum_{C(p,\xi)} w_{C(p,\xi)} = 2(-W_d + 2 \sum_{C \in \mathcal{D}(p)} w_C) = 2\iota_p \,.$$

We finally conclude that in all cases $k_p = \deg G_p = 2\iota_p$. By the choice of the homotopy h,

$$I(\mathcal{K}_p, \sigma_p; M) = k_p \cdot I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M) = 2\iota_p \cdot I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M).$$

We finish the proof by observing that

$$\begin{aligned} I(\mathcal{K}_p, \{p\} \times \mathbb{S}^1; M) &= I(h(\mathbb{S}^1 \times [0,1]), [p]; \mathbb{R}^2) = I(h(\mathbb{S}^1 \times [0,1]), [p] - [\infty]; \mathbb{R}^2) \\ &= -I(\partial h(\mathbb{S}^1 \times [0,1]), [p,\infty]; \mathbb{R}^2) = \mathrm{ind}_p(\Gamma). \end{aligned}$$

$\square$

4. Finite type invariants.

Finite type invariants generalize polynomial functions. This notion is based on the following classical theorem:

Theorem 4.1 (Frechet 1912)**.** *Given* $x_0, x_1^{\pm}, \dots, x_n^{\pm} \in \mathbb{R}$ *and an* n*-tuple* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, *let* $x_\varepsilon = x_0 + x_1^{\varepsilon_1} + \dots + x_n^{\varepsilon_n}$ *and* $|\varepsilon| = \prod_{i=1}^n \varepsilon_i$. *Then* C^0*-function* $f : \mathbb{R} \to \mathbb{R}$ *is a polynomial of degree less than* n, *iff*

$$\sum_{\varepsilon \in \{-1,1\}^n} (-1)^{|\varepsilon|} f(x_\varepsilon) = 0$$

for any choice of x_0 *and* $x_1^{\pm}, \dots, x_n^{\pm}$.

Finite type invariants are topological analogues of this definition. Corresponding theories are developed for a variety of objects: knots, 3-manifolds, plane curves, graphs, etc. (see [13] for a general theory of finite type invariants of cubic complexes). Let us briefly recall the main notions in the case of immersed curves in a punctured plane. Let $\mathfrak{S} \subset \mathbb{R}^2$ be a finite set of marked points and Γ_{sing} be an immersed plane curve with n non-generic fragments, contained in n small disks $\mathbb{D}_i$ (all in general position). Fix an arbitrary pair of resolutions for each $\mathbb{D}_i$ and call one of them positive and the other negative (again, arbitrarily). Here by a resolution of Γ_{sing} in a disk $\mathbb{D}_i$ we mean a homotopy of Γ_{sing} inside $\mathbb{D}_i$, fixed on the boundary $\partial\mathbb{D}_i$, so that the resulting curve is a generic immersion inside $\mathbb{D}_i$ and does not pass through $\mathfrak{S} \cap \mathbb{D}_i$. See Figure 14.

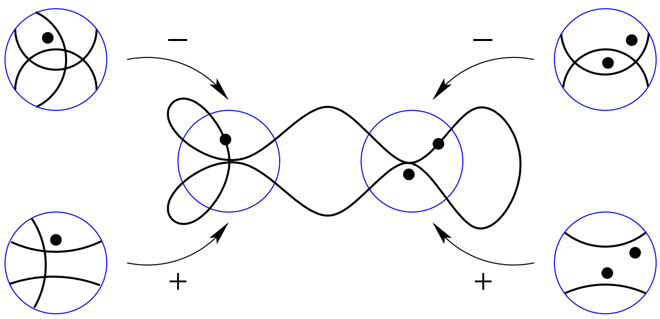

Figure 14. A non-generic curve with a pair of resolutions in each disk.

For an n-tuple $\varepsilon \in \{-1, 1\}^n$, resolve all singularities of Γ_{sing} choosing the corresponding ε_i resolution in each disk $\mathbb{D}_i$. Denote by Γ_ε the resulting curve. In this way, as ε runs over $\{-1, 1\}^n$, we obtain 2^n generically immersed curves Γ_ε. See Figure 15.

Denote $|\varepsilon| = \prod_{i=1}^n \varepsilon_i$. A locally-constant function f on the space of generically immersed curves is called an invariant of degree less than n, if

$$\sum_{\varepsilon \in \{-1,1\}^n} (-1)^{|\varepsilon|} f(\Gamma_\varepsilon) = 0,$$

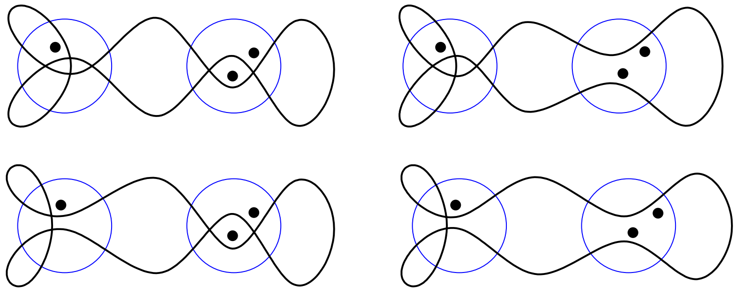

FIGURE 15. Resolved generic curves.

for any choice of the curve Γ_{sing} and its resolutions.

When $\mathfrak{S} = \varnothing$, the only invariant of degree zero (i.e., a constant function on the space of immersed curves) is the rotation number $\mathrm{ind}(\Gamma)$. Various interesting invariants of degree one for $\mathfrak{S} = \varnothing$ were extensively studied by V. Arnold, see [1]. When $\mathfrak{S}$ consists of one point, we get an additional simple invariant of degree one, namely $\mathrm{ind}_p(\Gamma)$. In a general case, any linear combination of $\mathrm{ind}(\Gamma)$ and $\mathrm{ind}_p(\Gamma)$, $p \in \mathfrak{S}$ is an invariant of degree at most one.

Finite type invariants naturally appear in real enumerative geometry. One of the simplest examples was considered in Section 1.2. Note that in the formula (1), an algebraic number of lines passing through a point p and tangent to a generic immersed curve $\Gamma \subset \mathbb{R}^2 \smallsetminus \{p\}$ is expressed via invariants $\mathrm{ind}(\Gamma)$, $\mathrm{ind}_p(\Gamma)$ of degrees zero and one. This fact is easy to explain. Let us show, that if a certain algebraic number of lines satisfying some passage/tangency conditions is a locally constant function f on the space of generic immersed curves, then it is an invariant of degree less than or equal to two. Indeed, let Γ_{sing} be an immersed curve with three non-generic fragments contained in three small disks $\mathbb{D}_i$, $i = 1, 2, 3$, which do not lie on one line (i.e., no line passes through all three of them). Suppose that some line l is counted for one of the resolutions Γ_ε of Γ_{sing}. Then l does not pass through at least one of the disks, say, $\mathbb{D}_1$. But then l is counted twice – with opposite signs – for both resolutions of Γ_{sing} inside $\mathbb{D}_1$, hence its contribution to f sums up to 0, and we readily get $f(\Gamma_{sing}) = 0$.

By the same argument (noticing that no rational curves of degree d pass through $3d$ generic points), we immediately obtain the following

Theorem 4.2. *Suppose that a certain algebraic number of real rational algebraic plane curves of degree d, satisfying some passage/tangency conditions, is a locally constant function on the space of generic immersed curves. Then it is an invariant of degree less than or equal to $3d - 1$.*

Moreover, if a curve is required to pass through k fixed points (in general position), then an algebraic number of such curves is an invariant of degree less than or equal to $3d - k - 1$. In particular, for $k = 3d - 2$ we get the upper bound one on the degree of an invariant. This explains the structure of formula (2) of Theorem 2.1.

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Department of Mathematics, Technion- Israel Institute of Technology, Haifa 32000, Israel

E-mail address: serjl@tx.technion.ac.il

Department of Mathematics, Technion- Israel Institute of Technology, Haifa 32000, Israel

E-mail address: polyak@math.technion.ac.il