Enumerative geometry and finite type invariants

or where rigid algebraic geometry meets smooth topology

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Rigidity vs. Flexibility

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- In complex geometry, objects are rigid (local behavior determines global).
- In smooth topology objects are flexible (any local changes are possible).
- Algebraic objects remain rigid even in real situation, so, to take the best of both smooth/algebraic worlds, we will take some objects rigid and some smooth.

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- In complex geometry, objects are counted ~~without signs~~ with positive signs.
- In real topology, objects are always counted with signs.

Generic vs. non-generic

Consider a space $\mathrm{Imm}(S^1, \mathbb{R}^2)$ of generic immersions of S^1 to $\mathbb{R}^2$:

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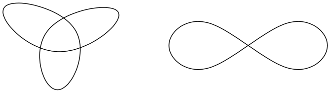

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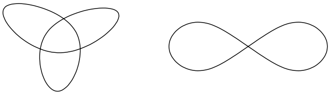
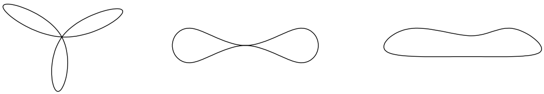


Generic immersions



Non-generic immersions

Counting tangents- a toy model

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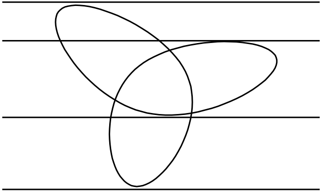
- We all know that there is one line passing through a point in a given direction.
- Now, how many lines are tangent to a generic (oriented) immersed curve and pass in a given direction?



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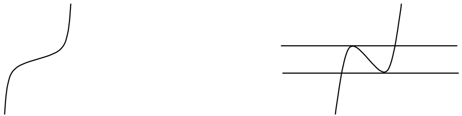
- In particular, it should be preserved under the following move:



No tangents — Two new tangents

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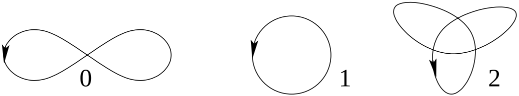
- Then the algebraic number of tangents is $2\mathrm{rot}C$, where $\mathrm{rot}C$ is the Whitneys rotation number, i.e. the number of turns made by a tangent vector as we pass once along the curve C:



- Now, lets count lines passing through a given point p and tangent to a generic (oriented) immersed curve C. Signs remain the same.

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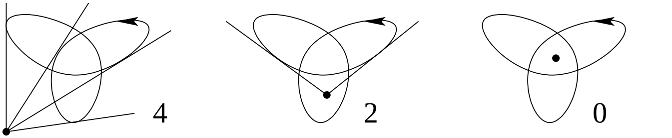
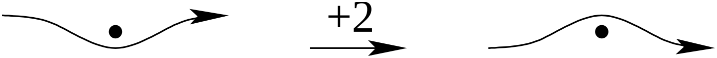
- The result is invariant under homotopies in $\mathrm{Imm}(S^1, \mathbb{R}^2 \smallsetminus p)$, but changes when a homotopy passes through the fixed point p:



- What kind of invariant is it? When the fixed point is far away from the curve, we again get $2\mathrm{rot}C$ (take p to infinity to see it). When a homotopy passes through p, the number of tangents jumps by ± 2:



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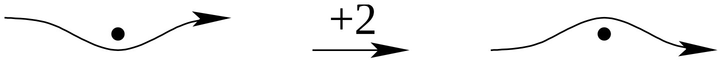
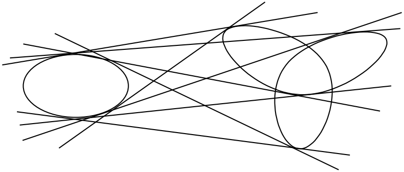


- So, while the number of lines changes, the jumps happen in a well-controlled way. In fact, it is easy to find a compensating term, and the formula becomes $2\mathrm{rot}C - 2\mathrm{ind}(p, C)$.

Counting double tangents

- Let's now count double tangents to a pair of immersed curves in $\mathbb{R}^2$:



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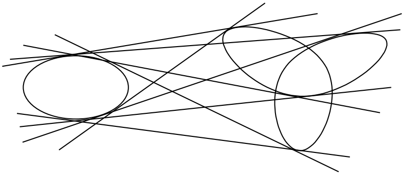
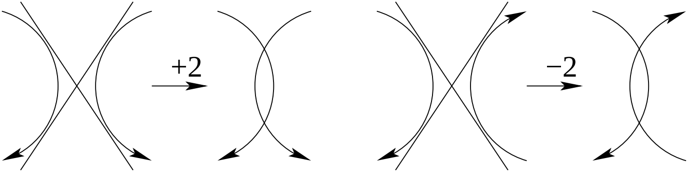


- We get the same sign rule (with the sign of a line being the product of signs of the two tangency points):

Then the algebraic number of double tangents for disjoint curves situated "far away" is $4 \operatorname{rot} C_1 \operatorname{rot} C_2$, and it jumps by -2 or $+2$ when curves experience a tangency:



Remark

In fact, we recover invariants J^+ and J^- introduced by Arnold (1992).

Generalize! Generalize!

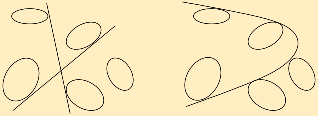
Instead of lines, we can consider higher degree curves, or hyperplanes etc. in similar enumerative problems. Even a simple knowledge that this can be done leads to non-trivial results! Indeed, suppose that we know that the invariant exists. What can we deduce from that?

Example of conics

Definition

We will say that a degenerate conic (i.e. a cross of lines) touches 5 generic ovals (= embedded curves), if it is tangent to 4 out of 5.

A non-degenerate conic touches the ovals if it is tangent to all of them:

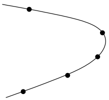


Theorem (Welschinger 2005)

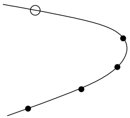
There are ≥ 272 real conics which touch 5 generic non-nested ovals. If the ovals are convex, at least 32 of these conics are non-degenerate.

Can easily generalize to conics touching 5 immersed curves and give a simple proof.

1/2 Proof: It is known that through 5 generic points there passes exactly one irreducible conic:



Take a very small circle centered in one of the points.



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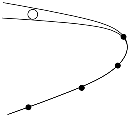

There are two conics passing through 4 other points and touching this circle (one on the outside and one on the inside).

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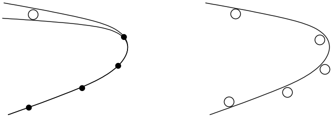


There are two conics passing through 4 other points and touching this circle (one on the outside and one on the inside).

Now take a very small circle with the center in the second point, etc. Finally, we get $2^5 = 32$ irreducible conics touching these circles.

Now lets compute the number of reducible conics. There are 3×5 ways to pick two pairs of circles out of 5 and 4 tangent lines to each pair of circles, thus we get a total of $240 = 3 \times 5 \times 4 \times 4$ reducible conics touching these circles.

Adding up, we get a total of $272 = 32 + 240$ conics. But our total algebraic number of conics is homotopy-invariant, so the answer for any 5 ovals is the same as for 5 small circles and we are done.

Main questions

In all of the above problems the same two questions arise:

1. How to introduce such signs, i.e. construct such invariants, in a simple and systematic manner?
2. How to identify the resulting invariants?

Q2: Which invariants appear?

It is easier to answer the 2nd question:

Theorem

All such invariants are of finite type. When we count rational curves of degree n, this is an invariant of degree $< 3n/2$.

Recall, that finite type invariants generalize polynomial functions; corresponding theories are developed for a variety of objects: knots, 3-manifolds, plane curves, graphs, etc.

This notion is based on the following.

Theorem (Frechet 1912)

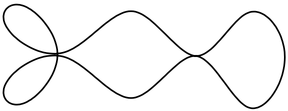
Given $x_0, x_1^{\pm}, \ldots, x_n^{\pm} \in \mathbb{R}$ and an n-tuple $\sigma \in \{-1,1\}^n$, let $x_\sigma = x_0 + x_1^{\sigma(1)} + \cdots + x_n^{\sigma(n)}$. Then C^0-function $v : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree less than n, iff

$$\sum_{\sigma \in \{-1,1\}^n} (-1)^{|\sigma|} f(x_\sigma) = 0$$

for any choice of x_0 and $x_1^{\pm}, \ldots, x_n^{\pm}$.

Finite type invariants are topological analogues of this theorem/definition.

In particular, let Γ be a plane curve with n non-generic fragments:



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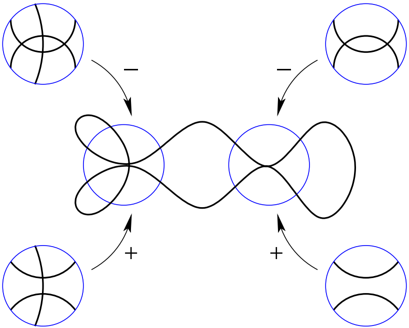
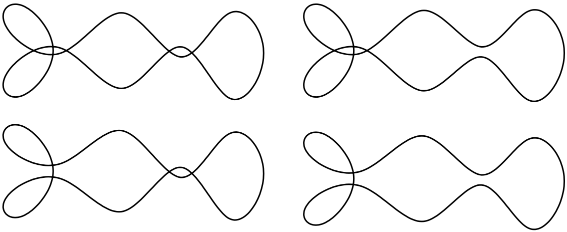


Fix a pair of resolutions for each fragment.

Choosing one of the two resolutions in each place, we get 2^n generic plane curves Γ_σ:



Definition

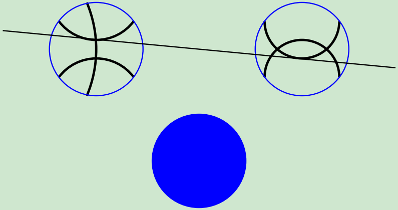
A homotopy-invariant function $v : \mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathbb{R}$ is a finite type invariant of degree $< n$, iff $\sum_\sigma (-1)^{|\sigma|} v(\Gamma_\sigma) = 0$ for any choice of Γ and $2n$ resolutions..

Example (Double tangents revisited)

Assume that (for a certain rule of signs) an algebraic number of double tangents is an invariant; let us prove that it is of degree < 3.
Pick an immersed curve with three non-generic singularities and fix two resolutions for each of them. We can make these in a small balls which do not lie on one line.
We are to prove that the corresponding alternating sum over all 2^3 resolved generic curves vanishes.

Example (Contd)

Any double tangent may involve fragments of resolutions near at most two singular points, thus it does not “notice” which resolution is chosen near the third singular point:



But then this double tangent is counted in the total alternating sum twice with opposite signs, so its contribution is $+1 - 1 = 0$.

Q1: How to find sign rules?

A1

Use maps of configuration spaces and homology intersections!

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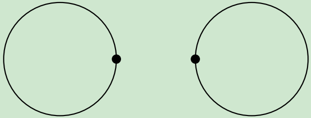
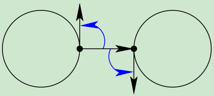
Example

For a pair of oriented immersed curves, let $\mathrm{Conf}_{1,1}$ be the space of (ordered) pairs of points:



Example (Contd)

Define $\phi : \mathrm{Conf}_{1,1} \to S^1 \times S^1$ by a pair of angles between the tangent vectors and the vector connecting the points:



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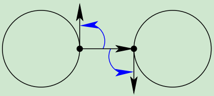

This is a map of two oriented closed manifolds of the same dimension, so has a well-defined degree. It can be calculated as an algebraic number of pre-images of any regular value.

Example (Contd)

In particular, one may count preimages of a value $(0, 0)$. But these are exactly double tangents of the type

Each preimage is counted with a sign (the local degree). These are just the signs we need!

Indeed, it is well-known that a degree is invariant under homotopy of a map.

Note BTW, that since the degree does not depend on a choice of regular value, we would get the same number for $(\pi, 0)$ which counts other double tangents, or for $(\pi/2, \pi/2)$ which counts double normals!

Other problems (number of circles touching three ovals, conics touching 5 ovals, etc. etc.) may be solved in the same way. E.g., for counting circles touching three ovals we consider a similar space $\mathrm{Conf}_{1,1,1}$ of triples and its map into $S^1 \times S^1 \times S^1$, etc.

There is another useful way to think about the meaning of such invariants. Namely, in a homological language the image $\phi(\mathrm{Conf}_{1,1})$ is a

2-cycle in $S^1 \times S^1$ and we intersect it with a point (s, s).

But we can also pick another map and intersect its image with some other cycle in the target!

For example, we can consider a similar map $\Phi : \mathrm{Conf}_{1,1} \to S^1 \times S^1 \times S^1$, mapping each pair to a pair of the tangent vectors and the vector connecting the points. Intersecting the image with the diagonal $\Delta = \{(x, x, x)\}$ gives the same answer.

Curves of higher degrees

Can use the same idea in higher degrees, e.g.

Theorem (S. Lanzat +M.P.)

The algebraic number of rational cubics passing through 7 generic points $P = \{p_1, \ldots, p_7\}$ and tangent to an immersed curve $C \in \mathbb{R}^2$ is
$$16\,(rotC - ind(P,C) + ind(R,C) - ind(Q,C))$$
Here R and Q are sets of nodes of reducible cubics and cusps of cuspidal cubics, respectively, passing through P.

Higher dimensions

Can use the same idea in higher dimensions, e.g.

Theorem (Arnold 1970's)

The algebraic number of planes in $\mathbb{R}^3$ which contain a fixed line l and are tangent to an immersed surface Σ is $\chi(\Sigma) - |l \cap \Sigma|$

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Similar formula hold for planes passing through a point and tangent to two immersed surfaces, or planes tangent to three immersed surfaces.

Higher dimensions- symplectic case

Consider a hypersurface Σ immersed in $\mathbb{R}^{2n}$ equipped with a standard symplectic form ω. Call a tangent to Σ isotropic, if it is in the kernel of the restriction of ω to $T\Sigma$. Let $p\mathbb{R}^{2n} \setminus \Sigma$ be a generic point.

Theorem

The algebraic number of isotropic tangents to Σ passing through p is $2\deg(G_\Sigma) - 2ind(p, S)$, where G_Σ is the Gauss map of Σ.

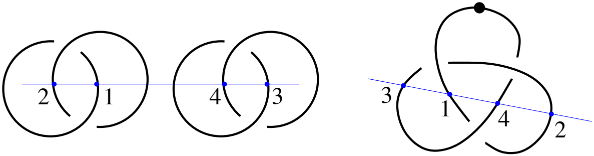
Knotty time!

A generic immersion of (several copies of) S^1 to R^3 is an embedding, so $\mathrm{Imm}(S^1, \mathbb{R}^3)$ is a space of generic knots (or links). Technically, it is more convenient to work with long (or string) links, i.e. embeddings of several copies of $\mathbb{R}^1$ to $\mathbb{R}^3$, which are standard outside a compact.

We can consider similar problems of intersection and/or tangency, counting various algebraic curves passing through/tangent to a link in several points.

Quadrisecants

E.g., we may count link/knot quadrisecants, i.e. lines intersecting a link (or a knot) in 4 points:



Note that for long knots and links, we may distinguish quadrisecants by the order of points along the line. Just as in the planar case, any algebraic number of various quadrisecants should be an invariant of degree < 3.

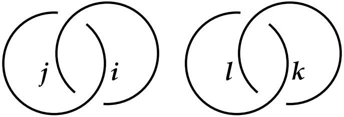
Theorem (J.Viro 2005)

The algebraic number of quadrisecants with a $(1,2,3,4)$ cyclic order for a 4-component link in $\mathbb{R}P^3$ is $2(lk_{12}lk_{34} - lk_{23}lk_{41})$

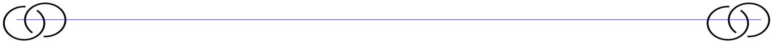
Theorem

The algebraic number of $(1,2,3,4)$ quadrisecants for a 4-component link in $\mathbb{R}^3$ is $lk_{12}lk_{34}$.

1/2 Proof: Suppose that we found a sign rule and constructed an invariant. Then it is easy to identify it with $\mathrm{lk}_{12}\mathrm{lk}_{34}$. Indeed, it is known that the only link invariants of degree 2 are linear combinations of $\mathrm{lk}_{ij}\mathrm{lk}_{kl}$ over transpositions $(ijkl)$ of (1234). Thus it remains to check its values on pairs of Hopf links:



It is an isotopy invariant, so we can take the pairs far apart and make them very small:



But then we just count lines in a fixed direction intersecting both components of each Hopf link- this is the corresponding linking number!

Theorem

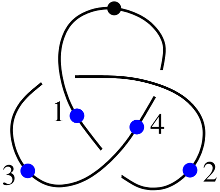
Counted with appropriate $(0, +1, -1)$- weights, the number of quadrisecants for a 3-component long link in $\mathbb{R}^3$ is the Milnors triple linking number μ_{123}

Theorem (Budney, Conant et.al. 2004)

Counted with appropriate signs, the number of (3,1,4,2) knot quadrisecants is v_2 (the 2-nd coefficient of the Alexander polynomial).

Proof takes 12 pages, with many technicalities involved. But this can be done quite easily!

Let K be a long knot. Define Conf_4 to be the space of (ordered) 4-tuples of points on K:



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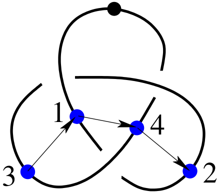

Consider a map $\phi : \mathrm{Conf}_4 \to S^2 \times S^2 \times S^2$, mapping each pair $(3, 1)$, $(1, 4)$, $(4, 2)$ to a direction of the vector connecting the points.

The image $\phi(\mathrm{Conf}_4)$ in $S^2 \times S^2 \times S^2$ intersects the diagonal $\Delta = \{(x, x, x)\}$ by points.

Theorem

$v_2 = I(\phi(Conf_4), \Delta)$

Conf_4 is an open space (an open 4-simplex). We compactify it in a standard naïve fashion, making it into a closed 4-simplex. It turns out that under knot isotopy $\phi(\partial \mathrm{Conf}_4)$ never intersects Δ, so $I(\phi(\mathrm{Conf}_4), \Delta)$ is an invariant. It is of degree ≤ 2; to identify it, compute it for the standard diagram of the trefoil.