From 3-manifolds to planar graphs and cycle-rooted trees

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"CONFIRMING THE BELIEF THAT MUSIC AND MATH ARE RELATED, I WILL NOW SING SOME LOVELY FRENCH EQUATIONS."

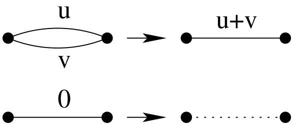
Outline

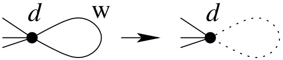
- Encode 3-manifolds by planar weighted graphs
- Pass from various presentations of 3-manifolds to graphs and back
- Similar encodings for related objects: links in 3-manifolds, manifolds with *Spin*- or $Spin^c$-structures, elements of the mapping class group, etc.
- Encoding is not unique: finite set of simple moves on graphs (related to electrical networks)
- Various invariants of 3-manifolds transform into combinatorial invariants
- Configuration space integrals $\rightarrow$ counting of subgraphs
- Low-degree invariants $\rightarrow$ counting of rooted forests

Chainmail graphs

A chainmail graph is a planar graph G, decorated with $\mathbb{Z}$-weights:

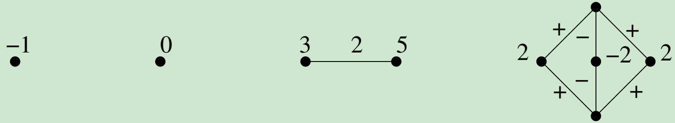
- Each vertex v is decorated with a weight $d(v)$; A vertex is balanced, if $d(v) = 0$ (can think about $d(v)$ as a "defect" of v); a graph is balanced, if all of its vertices are.
- Each edge e is decorated with a weight $w(e)$. A 0-weighted edge may be erased. Multiple edges are allowed. Two edges e_1, e_2 connecting the same pair of vertices may be redrawn as one edge of weight $w(e_1) + w(e_2)$. Looped edges are also allowed; a looped edge may be erased.





From graphs to manifolds

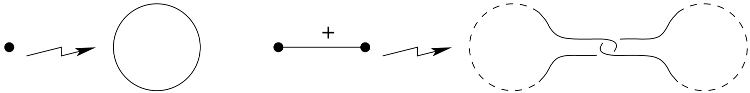
Example (Graphs, corresponding to some manifolds)



S^3 $\quad$ $S^2 \times S^1$ $\quad$ Poincare sphere $\quad$ $S^1 \times S^1 \times S^1$

Given a chainmail graph G with vertices v_i and edges e_{ij}, $i, j = 1, 2, \ldots, n$ we consruct a surgery link L as follows:

- vertex v_i $\rightarrow$ standard planar unknot L_i
- ± 1-weighted edge e_{ij} $\rightarrow$ ± 1-clasped ribbon linking L_i and L_j

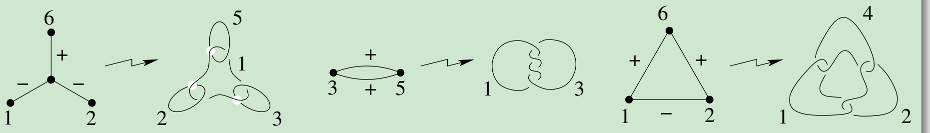


From graphs to manifolds

Linking numbers and framings of components are given by a graph Laplacian matrix Λ with entries

$$l_{ij} = \begin{cases} w_{ij}, & i \neq j \\ d_{ii} - \sum_{k=1}^{n} w_{ik} \ , & i = j \end{cases}$$

Example (Constructing a surgery link)



Different graphs and surgery links for the Poincare homology sphere

From manifolds to graphs

It turns out, that

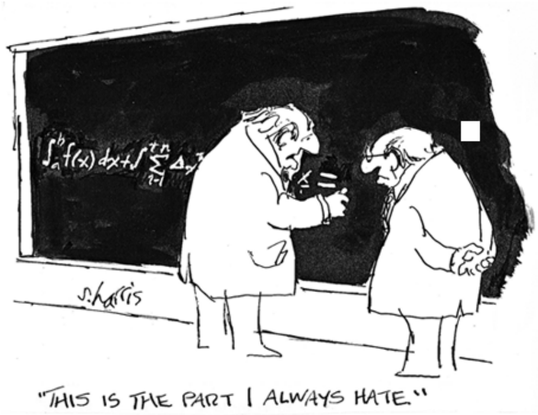
Theorem

Any (closed, oriented) 3-manifold can be encoded by a chainmail graph.

- Moreover, there are simple direct constructions starting from many different presentations of a manifold: surgery, Heegaard decompositions, plumbing, double covers of S^3 branched along a link, etc.
- Similar constructions work also for a variety of similar objects: links in 3-manifolds, 3-manifolds with *Spin*- or *Spin*^{c}-structures, elements of the mapping class group, etc.

Some info about M can be immediately extracted from G. In particular, M is a $\mathbb{Q}$-homology sphere iff $\det \Lambda \neq 0$ and then $|H_1(M)| = |\det \Lambda|$; also, signature of M is the signature $\mathrm{sign}(\Lambda)$ of Λ.

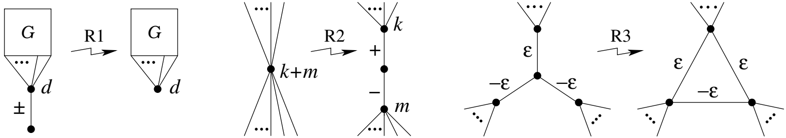
Proofs and explicit constructions ...



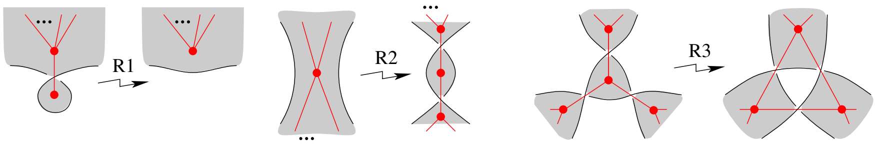
... No time to present here.

Calculus of chainmail graphs

An encoding of a manifold by a chainmail graph is non-unique. However, there is a finite set of simple moves which allow one to pass from one chainmail graph encoding a manifold to any other graph encoding the same manifold. The most interesting moves are



They are related to a number of topics: Kirby moves, relations in the mapping class group, electrical networks and cluster algebras, and Reidemeister moves for link diagrams (via balanced median graphs) -



Combinatorial invariants of 3-manifolds

Chern-Simons theory leads to a lot of knot and 3-manifold invariants. Attempts to understand the Jones polynomial in these terms led to quantum knot invariants, the Kontsevich integral, configuration space integrals and other constructions. In particular,

Perturbative CS-theory $\xrightarrow{\textit{Feynman diagrams}}$ Configuration space integrals

- Rather powerful: contain universal finite type invariants of knots and 3-manifolds
- Very complicated technically
- Extremely hard to compute

We expect a similar combinatorial setup in our case: An appropriate

CS-theory on graphs $\xrightarrow[\textit{Feynman diagrams}]{\textit{discrete}}$ Discrete sums over subgraphs

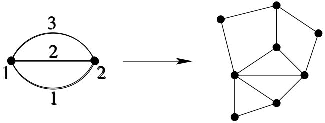
Types of subgraphs are suggested by the theory: uni-trivalent graphs for links; trivalent graphs for 3-manifolds.

Combinatorial invariants of 3-manifolds

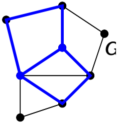
This actually works! Here is the setup: we pass from the manifold M to its combinatorial counter-part $\rightarrow$ a chainmail graph G. In both cases we use summations over similar Feynman graphs.

- Vertices of a Feynman graph:
  configurations of n points in $M \rightarrow$ sets of n vertices in G
- Edges of a Feynman graph:
  propagators in $M \rightarrow$ paths of edges in G
- Integration over the configuration space $\rightarrow$ sum over subgraphs
- Compactifications and anomalies due to collisions of points in $M \rightarrow$ appearance of degenerate graphs when several vertices merge together

Let's see this on an example of the simplest non-trivial perturbative invariant, corresponding to the Feynman graph with 2 vertices, i.e., the Θ-graph:

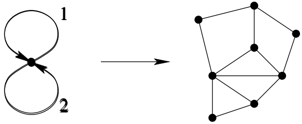


We count maps $\phi : \Theta \to G$ with weights and multiplicities. One can think about such a map as a choice of two vertices v_i and v_j of G, connected by 3 paths of edges which do not have any common internal vertices:

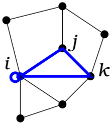


The weight $W(\phi)$ of ϕ is the product $L(\phi)\prod_{e\in\phi(G)} l_e$, where $L(\phi)$ is the minor of Λ, corresponding to all vertices of G not in $\phi(\Theta)$.

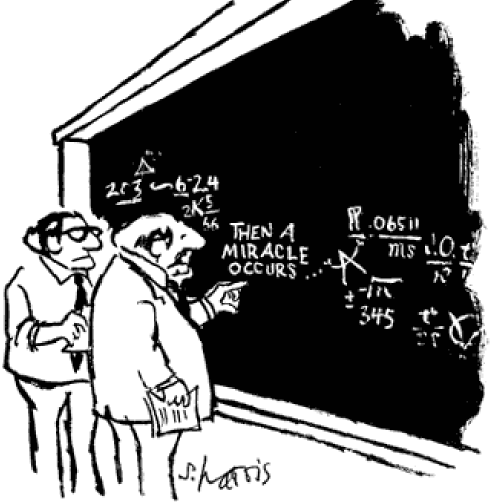
Degenerate maps should be counted as well. Such degeneracies appear when two vertices of the Θ-graph collide together to produce a figure-eight graph:



Diagonal entries of Λ also enter in the formula, when one lobe (or possibly both) of the figure-eight graph becomes a looped edge in the 4-valent vertex. The weight of such a loop in v_i is l_{ii}. E.g., for the map



we have $W(\phi) = L(\phi) \cdot l_{ij} \cdot l_{jk} \cdot l_{ki} \cdot l_{ii}$. In the most degenerate cases – a triple edge or double looped edge – weights need to be slightly adjusted.



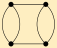
"I think you should be more explicit here in step two."

Θ-invariant of 3-manifolds

Theorem

$\Theta(G) = \sum_\phi W(\phi)$ is an invariant of M. If M is a $\mathbb{Q}$-homology sphere (i.e., $\det \Lambda \neq 0$), we have $\Theta(G) = \pm 12|H_1(M)|(\lambda_{CW}(M) - \frac{sign(M)}{4})$, where $\lambda_C W(M)$ is the Casson-Walker invariant.

Conjecture

The next perturbative invariant can be obtained in a similar way by counting maps of  *and*  *to G.*

Note that $\Theta(G)$ is a polynomial of degree $n+1$ in the entries of Λ. This leads to

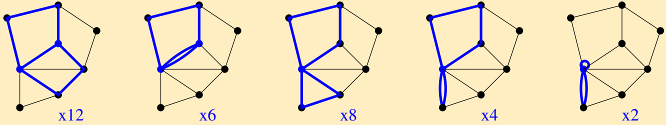
Conjecture

Any finite type invariant of degree d of 3-manifolds (with an appropriate normalization) is a polynomial of degree at most $n+d$ in the entries of Λ.

Θ-invariant of 3-manifolds

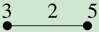
Remark

Instead of counting maps $\phi : \Theta \to G$, we may count Θ-subgraphs of G, taking symmetries into account:

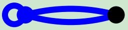
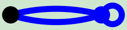


Example

For the (negatively oriented) Poincare homology sphere one has

$G =$ 

. Thus $\Lambda = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\det \Lambda = -1$ (so M is a $\mathbb{Z}$-homology sphere), $\mathrm{sign}(\Lambda) = 0$, and to compute $\Theta(G)$ we count

$2 \cdot ($  $+$  $) + ($  $+$  $) + 2 \cdot$  to get

$\Theta(G) = 2 \cdot (1 \cdot 2^2 + 3 \cdot 2^2) + (1^2 + 2)(-3) + (3^2 + 2)(-1) + 2 \cdot (2^3 - 2) = 24$ and obtain $\lambda_{CW}(M) = -2$.

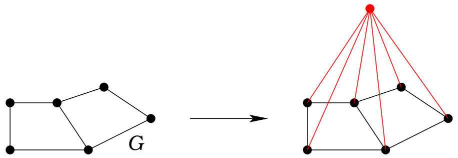
Counting cycle-rooted trees

Recall that the matrix Λ was defined as the graph Laplacian for the weight matrix W:

$$l_{ij} = \begin{cases} w_{ij}, & i \neq j \\ d_{ii} - \sum_{k=1}^{n} w_{ik} \ , & i = j \end{cases}$$

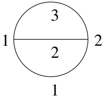
An expression for $\Theta(M)$ in terms of the original weight matrix W (with d_{ii} on the diagonal) is even simpler and can be achieved by a certain generalized version of the celebrated Matrix Tree Theorem.

For this purpose, we add to G a new balancing "super-vertex" v_0, connecting every vertex v_i of G to v_0 by an edge of the weight $w_{0i} = -d_{ii}$. We also change weights of all old vertices to 0 to get a balanced graph $\widehat{G}$:

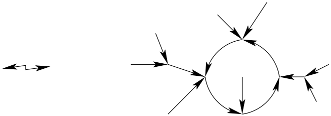


Counting cycle-rooted trees

The classical Matrix Tree Theorem states that $\det \Lambda$ equals to the weighted number of the spanning trees of $\widehat{G}$, where a tree T is counted with the weight $\prod_{e \in T} w(e)$.
It turns out, that one can pass from $\Theta(G)$ to a similar count of spanning cycle-rooted trees in $\widehat{G}$:



in G, with Λ-weights



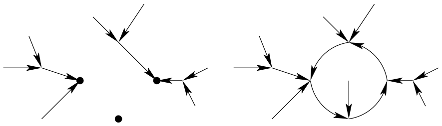
in $\widehat{G}$, with W-weights

This approach has a number of interesting applications and ramifications:

- Simpler computational formulas: no more degenerated cases, simpler graphs.
- Counting spanning cycle-rooted trees in $\widehat{G}$ to get $\Theta(G)$ leads to a new generalized version of the classical theorem Matrix Tree Theorem.

Cycle-rooted trees and orbits of vector fields

Finally, cycle-rooted trees can be interpreted as closed orbits of vector fields on a graph:
A discrete vector field on a graph is a choice of at most one outgoing edge at each vertex.



Critical vertices are those with no outgoing edges. An orbit may end in a critical point - these are trees with roots in critical points (and all edges oriented toward the root).
There are also closed orbits; these are cycle-rooted trees (with all edges oriented towards the cycle).
In these terms, the determinant $\det \Lambda$ counts vector fields on G with no closed orbits. The Θ-invariant counts vector fields with one closed orbit.

Cycle-rooted trees and orbits of vector fields

Time for speculations:
There is a highly suggestive continuous analogue for such a closed orbits counting: Gopakumar-Vafa's Gauge Theory/Geometry duality between the CS theory and closed strings on a resolved conifold. The closed strings theory suggested by Gopakumar-Vafa leads to a certain Floer-type symplectic homology setup.
It seems that in our discrete setting Gopakumar-Vafa duality boils down to the Laplace transform on graphs and corresponds to a generalized Matrix Tree Theorem.
We thus expect that there is a suitable chain complex and a homology theory in the cycle-rooted trees setup. Its construction is challenging.

"ON THE OTHER HAND, MY RESPONSIBILITY TO SOCIETY MAKES ME WANT TO STOP RIGHT HERE."