INTEGRATING CURVATURE: FROM UMLAUFSATZ TO J^+ INVARIANTS.

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Abstract. Hopf's Umlaufsatz relates the total curvature of a closed immersed plane curve to its rotation index. While the curvature of a curve changes under local deformations, its integral over a closed curve is invariant under regular homotopies. A natural question is whether one can find some natural densities on a curve, such that the corresponding integrals are (possibly after some corrections) also invariant under regular homotopies of the curve. We construct a family of such densities using indices of points relative to the curve. The corresponding generating function in a formal variable q may be considered as a quantization of the total curvature. The linear term in the Taylor expansion at $q = 1$ coincides, up to a normalization, with Arnold's J^+ invariant.

Let Γ be a closed oriented immersed plane curve $\Gamma : \mathbb{S}^1 \to \mathbb{R}^2$. One of the fundamental notions related to Γ is its *curvature* κ. Another important notion is that of a *rotation index* $\mathrm{rot}(\Gamma)$, i.e. the number of turns made by the tangent vector as we follow Γ along its orientation.

Hopf's Umlaufsatz [2] is one of the simplest versions of the Gauss-Bonnet theorem and one of the fundamental theorems in the theory of plane curves. It relates two different types of data: local geometric characteristic of a plane curve – its curvature κ – and a global topological characteristic – its rotation index $\mathrm{rot}(\Gamma)$. Although the curvature of a plane curve changes under local deformations, the theorem states that its average (integral) over a closed curve is invariant under homotopies in the class of immersed curves:

Theorem 1 (Hopf's Umlaufsatz)**.**

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} \kappa(t)\, dt = \mathrm{rot}(\Gamma) \tag{1}$$

A natural question is whether one can find some natural densities ρ on Γ such that the average $I_\rho(\Gamma) = \displaystyle\int_{\mathbb{S}^1} \kappa(t)\rho(t)\, dt$ is (possibly after

2010 *Mathematics Subject Classification.* 53A04, 57R42.

Key words and phrases. immersed plane curves, curvature, rotation number, regular homotopy.

some corrections) also invariant under local deformations of Γ. Since the rotation index is (up to normalization) the only invariant of Γ in the class of immersed curves, we cannot expect $I_\rho(\Gamma)$ to remain invariant under arbitrary homotopies. We can hope, however, that the result is invariant under regular homotopies. Here by a *regular homotopy* we mean homotopy in the class of *generic* immersions, i.e. immersions with a finite set X of transversal double points as the only singularities. Invariants of such a type were originally introduced by Arnold [1] and include the celebrated $J^\pm$ and St invariants (see [1] for details).

We construct a family of such densities using the *index* $\mathrm{ind}_p(\Gamma)$ *of* Γ *relative to a point* p. Given $p \in \mathbb{R}^2 \smallsetminus \Gamma$, we define $\mathrm{ind}_p(\Gamma)$ as the number of turns made by the vector pointing from p to $\Gamma(t)$, as we follow Γ along its orientation. This defines a locally-constant function on $\mathbb{R}^2 \smallsetminus \Gamma$. See Figure 1a. Suppose that Γ is generic. Then we can extend $\mathrm{ind}_p(\Gamma)$ to a $\frac{1}{2}\mathbb{Z}$-valued function on $\mathbb{R}^2$. To define $\mathrm{ind}_p(\Gamma)$ for $p \in \Gamma$, average its values on the regions adjacent to p – two regions if p is a regular point of Γ, and four regions if p is a double point of Γ. See Figure 1b. For each double

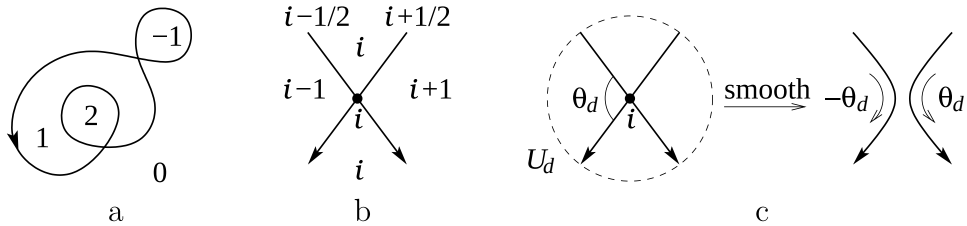

FIGURE 1. Indices of points and a smoothing of a double point.

point $d = \Gamma(t_1) = \Gamma(t_2) \in X$, define $\theta_d \in (0, \pi)$ as the (non-oriented) angle between two tangent vectors $\Gamma'(t_1)$ and $-\Gamma'(t_2)$. For $q \in \mathbb{R} \smallsetminus \{0\}$, define $I_q(\Gamma) \in \mathbb{R}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by

$$I_q(\Gamma) = \frac{1}{2\pi}\left(\int_{\mathbb{S}^1} \kappa(t) \cdot q^{\mathrm{ind}_{\Gamma(t)}(\Gamma)}\, dt - \sum_{d\in X} \theta_d \cdot q^{\mathrm{ind}_d(\Gamma)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right) \tag{2}$$

Theorem 2. *$I_q(\Gamma)$ is invariant under regular homotopies of Γ.*

Proof. Note that we can generalize all above notions and formulas to the case of a multi-component curve $\Gamma : \sqcup_n \mathbb{S}^1 \to \mathbb{R}^2$ by a summation of the appropriate indices over the components of Γ.

Let us smooth the original curve Γ in each double point respecting the orientation to get a multi-component curve $\widetilde{\Gamma} = \cup_n \widetilde{\Gamma}_n$ without double points. Then values of I_q on Γ and $\widetilde{\Gamma}$ differ by an easily computable factor, which depends only on the regular homotopy class of Γ. Indeed, consider a small neighborhood U_d of a double point d of index i, see Figure 1c. Under smoothing of d, the total curvature of $\widetilde{\Gamma} \cap U_d$ differs from that of $\Gamma \cap U_d$ by $\pm(\pi - \theta_d)$ for the fragment with index $i \pm \frac{1}{2}$, see Figure 1c. Thus the integral part of I_q changes by $\frac{1}{2\pi}(\pi - \theta_d)(q^{i+\frac{1}{2}} - q^{i-\frac{1}{2}})$. Also, the double point d contributes $-\frac{1}{2\pi}\theta_d q^i (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ to $I_q(\Gamma)$. Smoothing removes d, so this summand disappears from $I_q(\widetilde{\Gamma})$. Thus, the total change of I_q under smoothing of d equals $\frac{1}{2} q^i (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$. Hence

$$I_q(\Gamma) = I_q(\widetilde{\Gamma}) - \frac{1}{2} \sum_d q^{\mathrm{ind}_d(\Gamma)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\,.$$

Since $\sum_d q^{\mathrm{ind}_d(\Gamma)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ is invariant under regular homotopies of Γ, it remains to prove the invariance of $I_q(\widetilde{\Gamma}) = \sum_n I_q(\widetilde{\Gamma}_n)$.

Note that $\mathrm{ind}_{\widetilde{\Gamma}(t)}(\widetilde{\Gamma})$ is constant on each component $\widetilde{\Gamma}_n$ of $\widetilde{\Gamma}$, so

$$I_q(\widetilde{\Gamma}_n) = \frac{1}{2\pi} \int_{\mathbb{S}^1} \kappa_n(t) \cdot q^{\mathrm{ind}_{\widetilde{\Gamma}_n(t)}(\widetilde{\Gamma})}\, dt = q^{\mathrm{ind}_{\widetilde{\Gamma}_n(t)}(\widetilde{\Gamma})} \frac{1}{2\pi} \int_{\mathbb{S}^1} \kappa_n(t)\, dt$$

and by Umlaufsatz (1) we get $I_q(\widetilde{\Gamma}_n) = \pm\, q^{\mathrm{ind}_{\widetilde{\Gamma}_n(t)}(\widetilde{\Gamma})}$, depending on $\mathrm{rot}(\widetilde{\Gamma}_n) = \pm 1$. Thus, $I_q(\widetilde{\Gamma}_n)$ is invariant under regular homotopies of $\widetilde{\Gamma}$. But a regular homotopy of Γ induces a regular homotopy of $\widetilde{\Gamma}$ and the theorem follows.

□

Any two immersions with the same rotation number can be connected by regular homotopy and a finite sequence of self-tangency and triple-point modifications, shown in Figure 2. Depending on orientations and

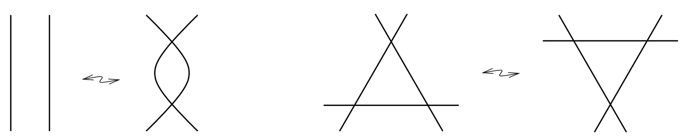

FIGURE 2. Self-tangency and triple-point modifications.

indices of adjacent regions, one can distinguish several types of these modifications. Self-tangencies can be separated into direct and opposite, shown in Figure 3a and 3b respectively. An index of a self-tangency modification is the index of two new-born double points (e.g., modifications in Figure 3 are of index i). Triple-point modifications can be separated into weak (or acyclic) and strong (or cyclic), shown in Figure 4a and 4b respectively. An index of a triple-point modification[1] is the minimum of indices of double points involved in this modification (e.g., modifications in Figure 4 are of index i).

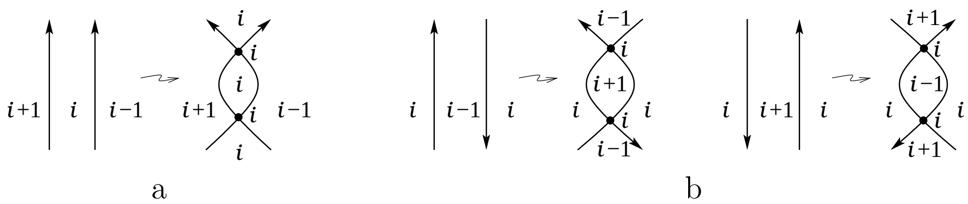

FIGURE 3. Direct and opposite self-tangency modifications of index i.

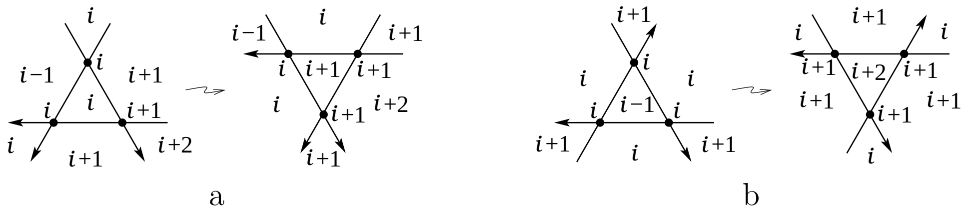

FIGURE 4. Weak and strong triple-point modifications of index i.

Invariants of regular homotopy are uniquely determined by their behavior under these modifications, together with normalizations on standard curves K_i of $\mathrm{rot}(K_i) = i$, $i = 0, \pm1, \pm2, \ldots$ shown in Figure 5. Basic invariants $J^\pm$ and St of (regular homotopy classes of) generic plane

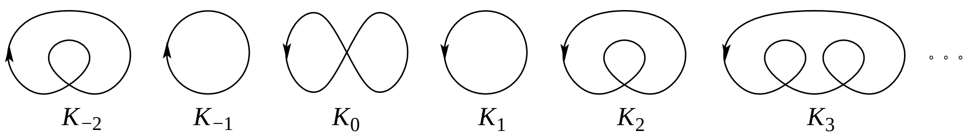

FIGURE 5. Standard curves of indices $0, \pm1, \pm2, \ldots$.

curves were introduced axiomatically by Arnold [1]. In particular, J^+ is uniquely determined by the following axioms:

- J^+ does not change under an opposite self-tangency or triple-point modifications.
- Under a direct self-tangency modification which increases the number of double points, J^+ jumps by 2.

[1]Our indices of modifications differ from the ones of [3] by an -1 shift.

- On the standard curves K_i we have $J^+(K_0) = 0$ and $J^+(K_i) = -2(|i| - 1)$ for $i = \pm 1, \pm 2, \ldots$.

In a similar way, $I_q(\Gamma)$ is uniquely determined by the following

Theorem 3. *The invariant $I_q(\Gamma)$ satisfies the following properties:*

- *$I_q(\Gamma)$ does not change under opposite self-tangencies.*
- *Under direct self-tangencies of index i, the invariant $I_q(\Gamma)$ jumps by $-q^i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$.*
- *Under (both weak and strong) triple-point modifications of index i, $I_q(\Gamma)$ jumps by $-\frac{1}{2}q^{i+\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$.*
- *We have $I_q(-\Gamma) = -I_{q^{-1}}(\Gamma)$, where $-\Gamma$ denotes Γ with the opposite orientation.*
- *On the standard curves K_i we have $I_q(K_0) = \dfrac{1}{2}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ and $I_q(K_i) = \dfrac{1}{2}(i-1)\, q^{\frac{3}{2}} + \dfrac{1}{2}(i+1)\, q^{\frac{1}{2}}$ for $i = 1, 2, \ldots$*

Proof. A straightforward computation verifies both the behavior of $I_q(\Gamma)$ under self-tangencies and triple-point modifications and its values on the curves K_i. To verify the behavior of $I_q(\Gamma)$ under an orientation reversal, note that $\operatorname{ind}_p(-\Gamma) = -\operatorname{ind}_p(\Gamma)$, which corresponds to the involution $q \to q^{-1}$ in terms $q^{\operatorname{ind}_{\Gamma(t)}(\Gamma)}$ and $q^{\operatorname{ind}_d(\Gamma)}$ of (2). Also, both terms in (2) change signs: the integral due to the change of parametrization, and the sum over double points due to the equality $q^{\frac{1}{2}} - q^{-\frac{1}{2}} = -\left((q^{-1})^{\frac{1}{2}} - (q^{-1})^{-\frac{1}{2}}\right)$. □

Substituting $q = 1$ into (2), we readily obtain $I_1(\Gamma) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \kappa(t)\, dt = \operatorname{rot}(\Gamma)$ and recover the classical Hopf Umlaufsatz, see Theorem 1. In this sense, invariant I_q may be considered as a quantization of the total curvature (1). Let us study the next term $I_1'(\Gamma)$ of the Taylor expansion of $I_q(\Gamma)$ at $q = 1$. From (2) we immediately get

$$I_1'(\Gamma) = \frac{1}{2\pi}\left(\int_{\mathbb{S}^1} \kappa(t)\cdot \operatorname{ind}_{\Gamma(t)}(\Gamma)\, dt - \sum_{d\in X} \theta_d\right).$$

Proposition 4. *$I_1'(\Gamma)$ can be identified with Arnold's J^+ invariant via $I_1'(\Gamma) = \frac{1}{2}(1 - J^+(\Gamma))$.*

Proof. Indeed, note that by Theorem 2, $I_1'(\Gamma)$ is invariant under homotopies of Γ in the class of generic immersions. Differentiating at $q = 1$ expressions for jumps of $I_q(\Gamma)$ in Theorem 3 we immediately conclude that $I_1'(\Gamma)$ is invariant under opposite tangencies and triple-point modifications. Moreover, under direct tangencies, $I_1'(\Gamma)$ jumps by -1. Thus its behavior under all modifications is the same as that of $-\frac{1}{2}J^+(\Gamma)$ (up to an additive constant depending on $\mathrm{rot}(\Gamma)$). A straightforward computation shows that $I_1'(\Gamma)$ takes values $I_1'(K_0) = \frac{1}{2}$ and $I_1'(K_i) = |i| - \frac{1}{2}$ for $i = \pm 1, \pm 2, \ldots$ on the standard curves K_i. Thus the proposition follows. □

Remark 5. *An infinite family of invariants, called "momenta of index" M_r together with their generating function $P_\Gamma(q) \in \mathbb{Z}[q, q^{-1}]$ were introduced by Viro in* [3, Section 5]*. A careful check of their behavior under self-tangencies and triple-point modifications, together with their values on the standard curves K_i, allow one to relate $P_\Gamma(q)$ to $I_q(\Gamma)$ as follows:*

$$P_\Gamma(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) I_q(\Gamma) + 1 + \frac{1}{2} \sum_{d \in X} q^{\mathrm{ind}_d(\Gamma)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$$

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