Plan: a Heegard splitting of a 3-mfd M^3
$\longrightarrow$ word in generators of the mapping class gp

$\rightsquigarrow$

planar graph with weights on edges
[M^3 can be easily reconstucted]

$\rightsquigarrow$

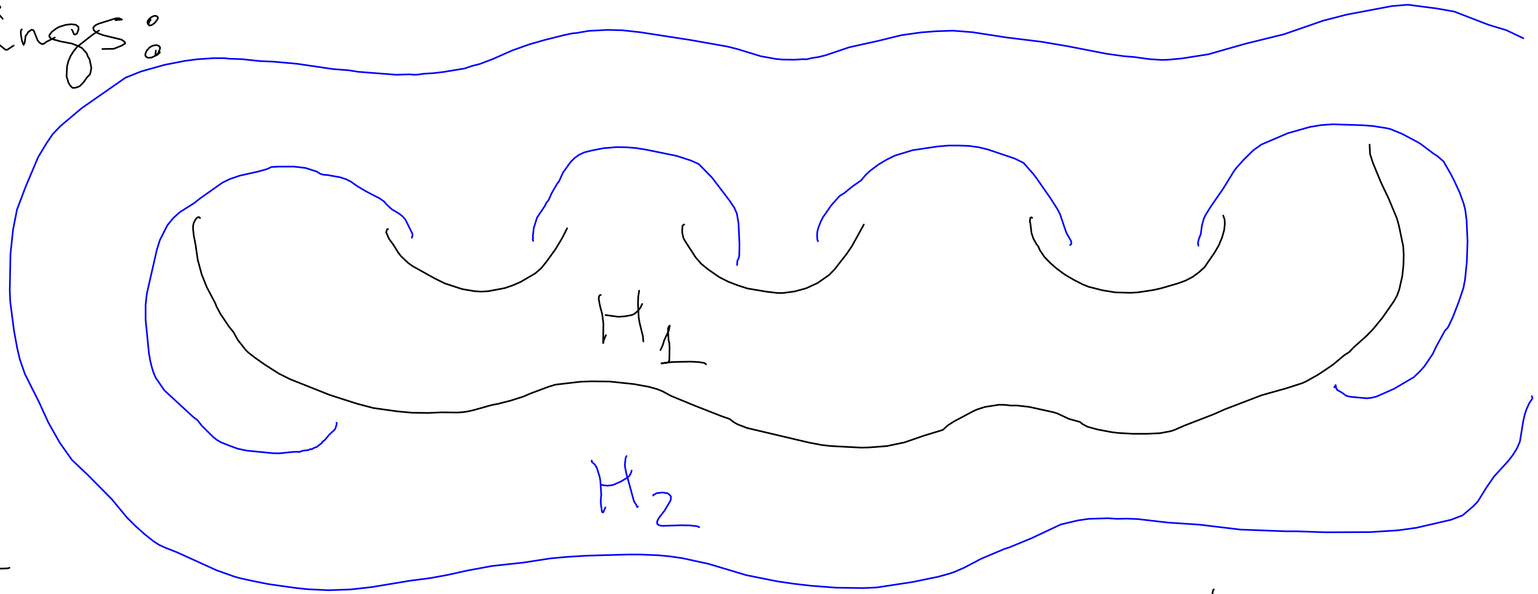
Invts of M^3 [Precise formulas for $|H_1(M^3)|$, the Casson-Walker invt; conjectural formulas for higher-degree FTI]. An alternative defn of FTI of 3-mfds.

Heegard splittings:

$M^3 = H_1 \cup_h H_2$

$h: \Sigma_g \circlearrowleft$

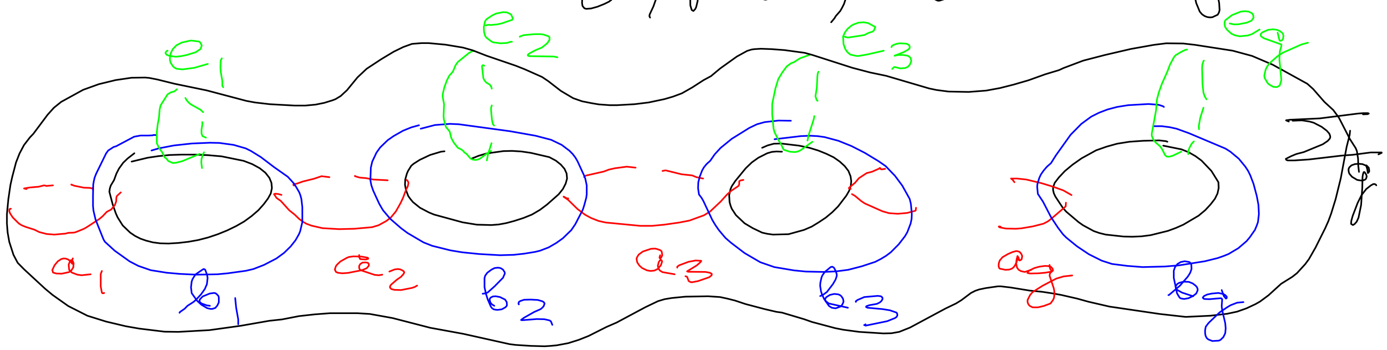
$\Sigma_g = \partial H_1 = \partial H_2$



Generators: Dehn twists $\alpha_i, \beta_i, \varepsilon_i$ along the curves

a_i, b_i, e_i

$i = 1, 2, \dots g$



$\underline{M^3}$ is given by a word w in $\alpha_i^{\pm 1}, \beta_i^{\pm 1}, \varepsilon_i^{\pm 1}$,

up to:
- (de) stabilization
- diffeo h_1, h_2 which extend to H_1, H_2

  $h_1 w h_2 \Longleftrightarrow w$
- relations in the mapping class gp.

  in particular,
$$\boxed{\begin{aligned} \alpha_i \beta_i \alpha_i &= \beta_i \alpha_i \beta_i \\ \alpha_{i+1} \beta_i \alpha_{i+1} &= \beta_i \alpha_{i+1} \beta_i \end{aligned}}$$

We will construct a planar weighted graph $G(w)$ and count certain subgraphs, so that it will be invt under all these moves $\to$ invt of M^3

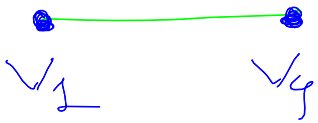
vertices of G: subwords of w, starting from $\beta_i^{\pm 1}$ until the next occurence of (the same) $\beta_i^{\pm 1}$

Ex: $w = \varepsilon_2 \alpha_1^{-1} \beta_1^{-1} \varepsilon_1 \varepsilon_3 \beta_2^{-1} \alpha_2^{-1} \beta_2 \alpha_2^{-1} \beta_1 \varepsilon_2 \varepsilon_2 \alpha_3$

edges of G: intersections of 2 subwords in β_i, β_j s.t. $j = i, i \pm 1$:

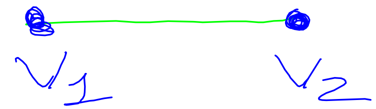
$$\underbrace{\beta_1^{-1} \varepsilon_1 \varepsilon_3 \overbrace{\beta_2^{-1} \alpha_2^{-1} \beta_2}^{v_3} \alpha_2^{-1} \beta_1}_{v_1}$$

Ex: $\beta_1^{-1} \varepsilon_1 \varepsilon_3 \beta_2^{-1} \alpha_2^{-1} \beta_2 \alpha_2^{-1} \beta_1 \varepsilon_2 \varepsilon_2 \alpha_3$

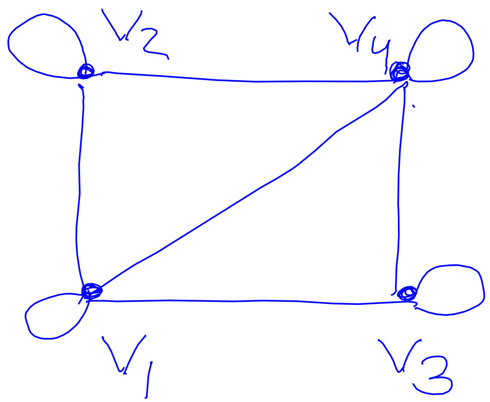


Rem: V_1 and V_2 (and also V_3, V_4) intersect at their ends -

$$\underbrace{\beta_1^{-1}\varepsilon_1\varepsilon_3\beta_2^{-1}\alpha_2^{-1}\beta_2\alpha_2^{-1}\beta_1}_{V_1}\underbrace{\varepsilon_2\varepsilon_2\alpha_3}_{V_2}$$



Ex $W = \varepsilon_2\alpha_1^{-1}\underbrace{\beta_1^{-1}\varepsilon_1\varepsilon_3\overbrace{\beta_2^{-1}\alpha_2^{-1}}^{V_3}\overbrace{\beta_2\alpha_2^{-1}\beta_1}^{}}_{V_1}\overbrace{\underbrace{\varepsilon_2\varepsilon_2\alpha_3}_{V_2}}^{V_4}$

$G(W) =$ 

This is a planar graph!

[place all vertices corresp. to subwords in β_i in the i^{th} column]

$\boxed{\text{Weights on edges}}$: 1 $\quad \underbrace{\beta_i^{\pm 1} - - - \overbrace{\beta_{i+1}^{\pm 1} - \beta_i^{\pm 1}}_{V'}}\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\overbrace{\phantom{\beta_{i+1}^{\pm 1} - \beta_i^{\pm 1}} - - - \beta_{i+1}^{\pm 1}}^{V''}$

count all α_i's with weight -1 (and α_i^{-1} with $+1$)

2 $\quad \underbrace{\beta_i^{\pm 1} - - - - \beta_i^{\pm 1}}_{V'}\underbrace{- - - \beta_i^{\pm 1}}_{V''}$ or $\underbrace{\beta_i^{\pm 1} - - - - - - \beta_i^{\pm 1}}_{V' = V''}$

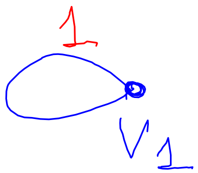
count all $\alpha_i, \alpha_{i+1}, \varepsilon_i, \beta_i$ with weights

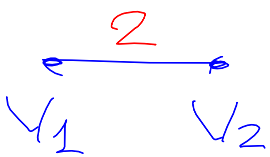
$\alpha_i, \alpha_{i+1} \rightsquigarrow +1, \quad \varepsilon_i \rightsquigarrow +1, \quad \beta_i \rightsquigarrow -1.$

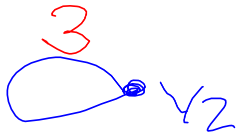
<u>Rem</u>

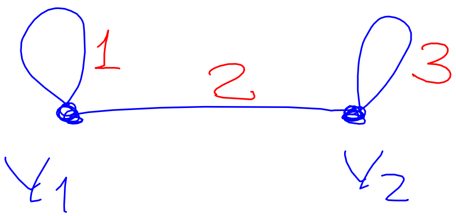
Remove edges of 0 weight.

Ex: $w = \varepsilon_1 \varepsilon_3 \varepsilon_3 \alpha_1 \underbrace{\beta_1^{-1} \overbrace{\beta_2^{-1} \varepsilon_2 \alpha_2^{-1} \varepsilon_1 \varepsilon_1 \alpha_2^{-1} \alpha_3 \varepsilon_2 \varepsilon_2}^{v_2}}_{v_1}$

looped edge  : $\overset{1}{\beta_1^{-1}} \beta_2^{-1} \varepsilon_2 \overset{-1}{\alpha_2^{-1}} \overset{1}{\varepsilon_1} \overset{1}{\varepsilon_1} \overset{-1}{\alpha_2^{-1}} \alpha_3 \varepsilon_2 \varepsilon_2$

edge  : $\beta_2^{-1} \varepsilon_2 \overset{1}{\alpha_2^{-1}} \varepsilon_1 \varepsilon_1 \overset{1}{\alpha_2^{-1}} \alpha_3 \varepsilon_2 \varepsilon_2$

looped edge  : $\overset{1}{\beta_2^{-1}} \overset{1}{\varepsilon_2} \overset{-1}{\alpha_2^{-1}} \varepsilon_1 \varepsilon_1 \overset{-1}{\alpha_2^{-1}} \overset{1}{\alpha_3} \overset{1}{\varepsilon_2} \overset{1}{\varepsilon_2}$

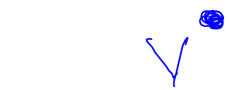
$G(w) =$ 

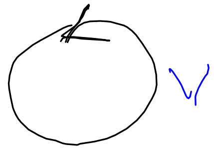
Adjacency matrix $\mathbb{L}_G = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

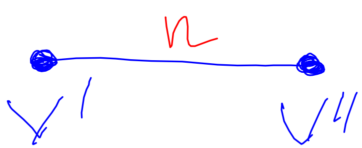
$D_G = \det(\mathbb{L}_G)$ $\qquad \sigma_G = \mathrm{signature}(\mathbb{L}_G)$

M^3 can be reconstructed from $G(M)$:

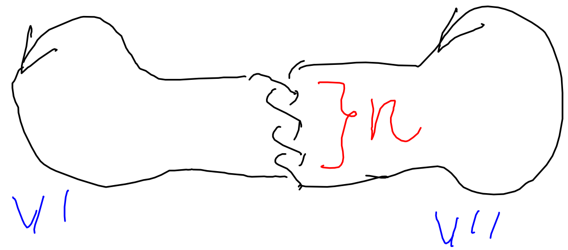
$G(M) \rightsquigarrow$ framed link $L \rightsquigarrow M^3 = S^3_L$

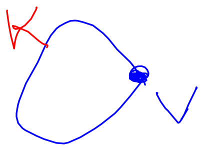


$\rightsquigarrow$ unknotted 

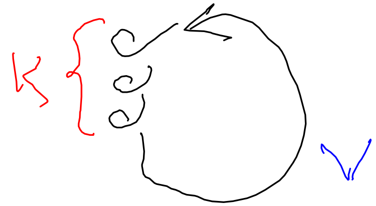


$\rightsquigarrow$ n clasps of the corresponding Os

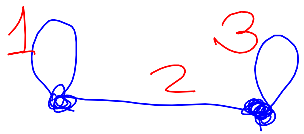




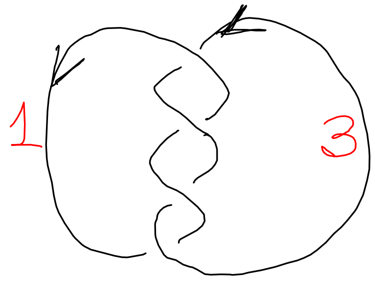
$\rightsquigarrow$ framing K



Ex

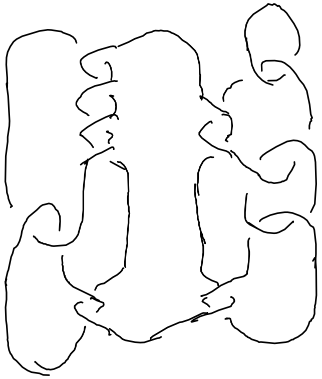


$\rightsquigarrow$



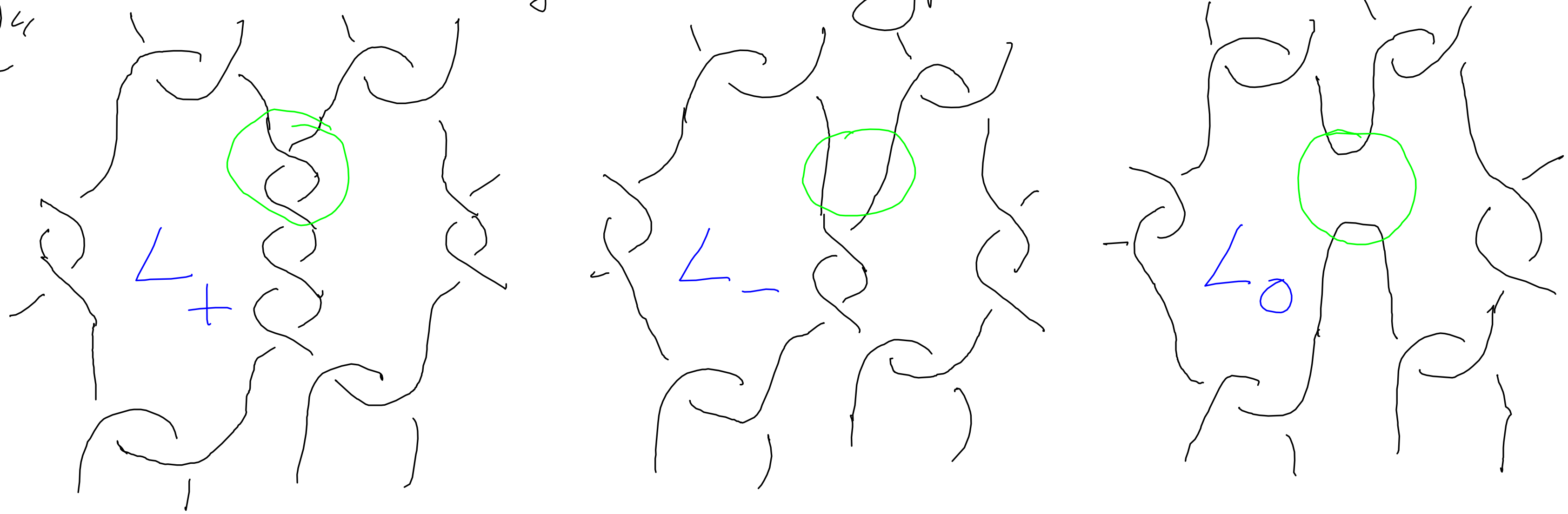
Thus (corollary of Matveev-M.P. 1992) $M^3 = S^3_L$

The "chain-mail" link L has some excellent properties;



"chain-mail"

in particular, it is easy to calculate its HOMFLY – all links in the skein rel are of this type:



The Alexander-Conway polynomial of $\mathcal{L}$ contains only the lowest degree term (determined by the linking numbers) all higher coeff. vanish.

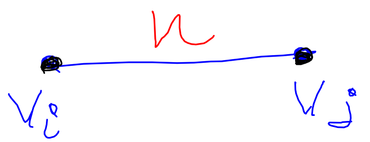
$$D_G = \det(\mathbb{L}_G) \text{ is } \begin{cases} \pm|H_1(M^3)| & \text{if } M \text{ is a } \mathbb{Q}HS \\ 0 & \text{otherwise} \end{cases}$$

σ_G = signature $(\mathbb{L}_G)$ is the signature of M^3

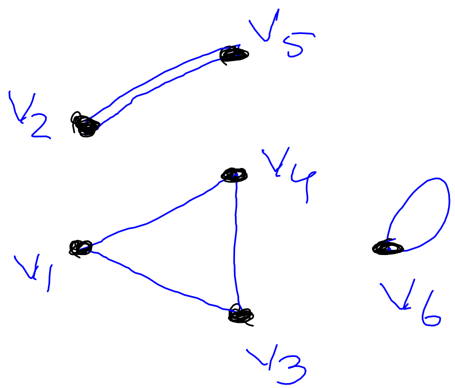
To explain how we can get λ_w, let us start from a graphical interpretation of D_G.

monomials in $\det(\mathbb{L}) \xleftrightarrow{1\text{-to-}1}$ collections of cycles in G (containing all vert.)

$l_{ij} = n \longleftrightarrow$

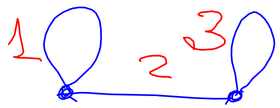


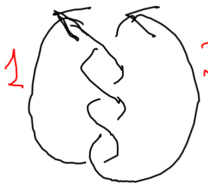
$l_{13}\, l_{34}\, l_{41}\, l_{25}\, l_{52}\, l_{66} \longleftrightarrow$



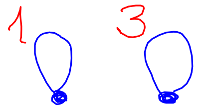
sign of a monomial $\longleftrightarrow (-1)^{\#\text{vert} - \#\text{cycles}}$

Ex $w = \varepsilon_1 \varepsilon_3 \varepsilon_3 \alpha_1 \beta_1^{-1} \beta_2^{-1} \varepsilon_2 \alpha_2^{-1} \varepsilon_1 \varepsilon_1 \alpha_2^{-1} \alpha_3 \varepsilon_2 \varepsilon_2$

$G =$ 

$\angle_G =$ 

$\angle\!\angle_G = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

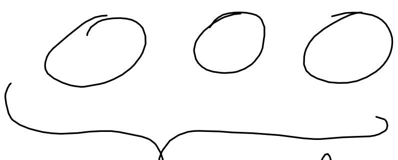
$D_G =$ 

$-$ 

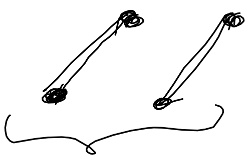
$= 1 \cdot 3 - 2^2 = -1$

$M^3 = S^3_L =$ Poincaré homology sphere (negatively oriented)

Conclusion: To calculate D_G, count embeddings in G of



collection of cycles



"degenerate cycles" - double edges

$\boxed{\text{weight of an embedding } \varphi = (-1)^{\#\text{vert}(G) - \#\text{cycles}} \cdot \prod_{e \in \varphi} \text{weight}(e)}$

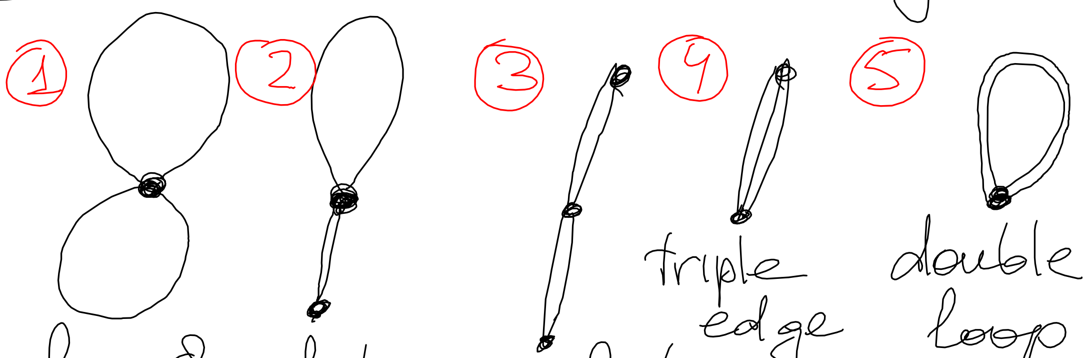
One can deal with λ_{xy} in a similar way: define

Θ = count embeddings in G of:

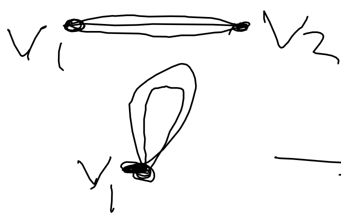
Θ-graph, collection of cycles, degenerate cycles

with weight $(-1)^{\#\text{vert}\,G \,-\, \#\text{cycles}} \cdot \prod_{e\in\varphi} \text{weight}(e)$

NB: should also count "degenerate Θ-graphs":



slightly different weights for



$\longrightarrow (\ell_{12}^3 - \ell_{12})$

$\longrightarrow (\ell_{11}^2 + 2)$

Also, should take into account symmetries of the subgraphs. Explicit formulas for #vert$G = 1, 2, 3, \dots$:

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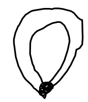
1 #vert $= 1$ $\qquad \mathbb{L} = (\ell_{11})$ $\qquad \Theta =$  $= \ell_{11}^2 + 2$

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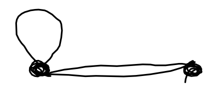
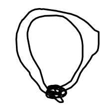
2 #vert $= 2$ $\qquad \mathbb{L} = \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix}$ $\qquad \Theta = 2$  $+ 2$  $-$   $=$

$$= 2(\ell_{12}^3 - \ell_{12}) + 2(\ell_{11} + \ell_{22})\ell_{12}^2 - (\ell_{11}^2 + 2)\ell_{22} - \ell_{11}(\ell_{22}^2 + 2)$$

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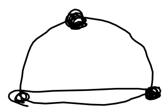
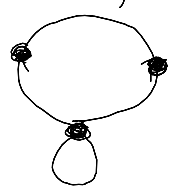
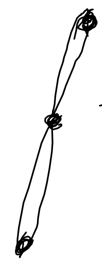
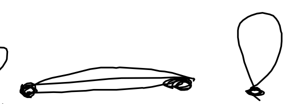
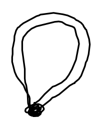
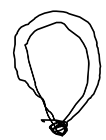
3 #vert $= 3$ $\qquad \Theta = -6$  -4  -2  $+2$ 

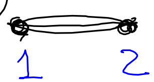
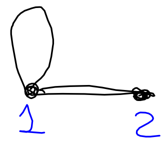
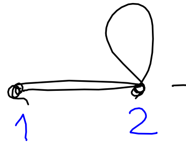
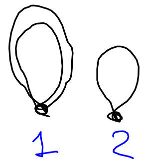
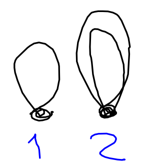
$$\mathbb{L} = \begin{pmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix}$$

$+ 2$   $-$    $+$  

Thm Let $M^3 = H_1 \cup_h H_2$ be a QHS.

Then $\Theta = 12 D_G \cdot \left(\lambda_w(M^3) - \frac{\sigma_G}{4} \right)$

$\pm |H_1(M^3)|$ Casson-Walker invt of M^3 signature of M

Ex $G =$  $\Theta = 2$  $+ 2$  $+ 2$  $-$  $-$ 

$D = -1$ $= 2(2^3-2) + 2\cdot 1 \cdot 2^2 + 2 \cdot 2^2 \cdot 3 - (1^2+2)\cdot 3 - 1\cdot(3^2+2)$

$\sigma = 0$ $= 24 \Rightarrow \boxed{\lambda_w = -2}$ ← Poincaré homology sphere

Two ways to prove the theorem:

(1) Use known results: Lescop surgery formula

$$12D\left(\lambda_W - \frac{\sigma}{4}\right) = \cancel{} + \text{combinatorial correction terms}$$

main part of the formula, involving derivatives of the Alexander polynomial for various sublinks of the surgery link

For our link all this part $= 0$!

So, we are left only with the correction terms in the Lescop formula — these are exactly the ones I denoted by Θ before.

② Can prove the theorem by an elementary combinatorial check (without Lescop results).

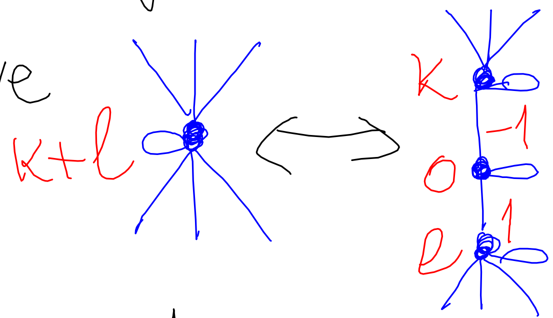
[This has an additional advantage — we can extend the defn of λ_w to mapping cylinders].

Invariance under (de)stabilization is immediate.

Invariance under $w \longleftrightarrow h_1 w h_2$ for homeos h_1, h_2 which extend to inrernal and external solid handlebodies H_1 and H_2 is also simple (the main part is hidden in our construction of the graph G and follows from Matveev–M.P. '92).

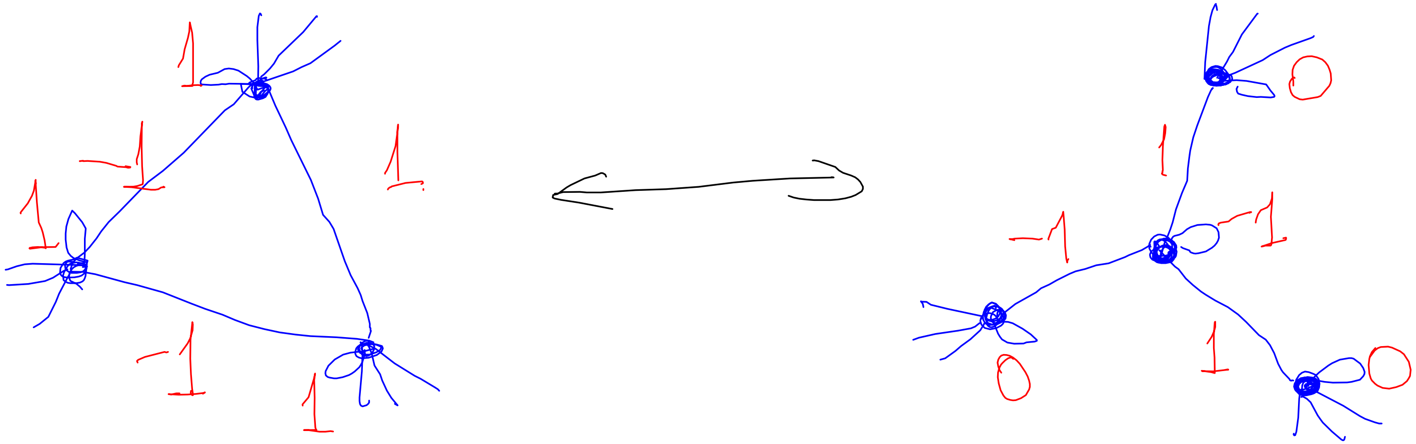
The most non-trivial part is to check the invariance under all rels of the mapping class group.

A simple move



(corresponding to an insertion of $\beta_i \beta_i^{-1}$ in w) allows to simplify G.

The relation $\alpha_i \beta_i \alpha_i = \beta_i \alpha_i \beta_i$ then boils down to a check of invariance of Θ under a version of a famous "star-triangle" equality:



- a careful counting of all subgraphs entering in Θ (including all degenerate cases) provides the result.